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# The recognition of triangle graphs☆

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Trapezoid graphs are the intersection graphs of trapezoids, where every trapezoid has a pair of opposite sides lying on two parallel lines $L_1$ and $L_2$ of the plane. This subclass of perfect graphs has received considerable attention as it generalizes in a natural way both interval and permutation graphs. In particular, trapezoid graphs have been introduced in order to generalize some well known applications of these graphs on channel routing in integrated circuits. Strictly between permutation and trapezoid graphs lie the *triangle* graphs – also known as *PI** graphs (for Point-Interval) – where the intersecting objects are triangles with one point of the triangle on the one line and the other two points (i.e. interval) of the triangle on the other line. Note that there is no restriction on which line between $L_1$ and $L_2$ contains one point of the triangle and which line contains the other two. Due to both their interesting structure and their practical applications, several efficient algorithms for optimization problems that are NP-hard in general graphs have been designed for trapezoid graphs – which also apply to triangle graphs. In spite of this, the complexity status of the triangle graph recognition problem (namely, the problem of deciding whether a given graph is a triangle graph) has been the most fundamental open problem on this class of graphs since its introduction two decades ago. Moreover, since triangle graphs lie naturally between permutation and trapezoid graphs, and since they share a very similar structure with them, it was expected that the recognition of triangle graphs is polynomial, as it is also the case for permutation and trapezoid graphs. In this article we surprisingly prove that the recognition of triangle graphs is NP-complete, even in the case where the input graph is known to be a trapezoid graph.



## 1. Introduction

A graph $G = (V, E)$ with $n$ vertices is the *intersection graph* of a family $F = \{S_1, \ldots, S_n\}$ of subsets of a set $S$ if there exists a bijection $\mu : V \to F$ such that for any two distinct vertices $u, v \in V$, $uv \in E$ if and only if $\mu(u) \cap \mu(v) \neq \emptyset$. Then, $F$ is called an *intersection model* of $G$. Note that every graph has a trivial intersection model based on adjacency relations [18]. However, some intersection models provide a natural and intuitive understanding of the structure of a class of graphs, and turn out to be very helpful to obtain structural results, as well as to find efficient algorithms to solve optimization problems [18]. Many important graph classes can be described as intersection graphs of set families that are derived from some kind of geometric configuration.

Consider two parallel horizontal lines on the plane, $L_1$ (the upper line) and $L_2$ (the lower line). Various intersection graphs can be defined on objects formed with respect to these two lines. In particular, for *permutation* graphs, the objects are line segments that have one endpoint on $L_1$ and the other one on $L_2$. Generalizing to objects that are trapezoids with one interval

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on $L_1$ and the opposite interval on $L_2$, *trapezoid* graphs have been introduced independently in [5], [6]. Given a trapezoid graph $G$, an intersection model of $G$ with trapezoids between $L_1$ and $L_2$ is called a *trapezoid representation* of $G$. Trapezoid graphs are perfect graphs [3,9] and generalize in a natural way both interval graphs (when the trapezoids are rectangles) and permutation graphs (when the trapezoids are trivial, i.e. lines). In particular, the main motivation for the introduction of trapezoid graphs was to generalize some well known applications of interval and permutation graphs on channel routing in integrated circuits [6].

Moreover, two interesting subclasses of trapezoid graphs have been introduced in [5]. A trapezoid graph $G$ is a *simple-triangle* graph if it admits a trapezoid representation, in which every trapezoid is a triangle with one point on $L_1$ and the other two points (i.e. interval) on $L_2$. Similarly, $G$ is a *triangle* graph if it admits a trapezoid representation, in which every trapezoid is a triangle, but now there is no restriction on which line between $L_1$ and $L_2$ contains one point of the triangle and which one contains the other two points (i.e. the interval) of the triangle. Such an intersection model of a simple-triangle (resp. triangle) graph $G$ with triangles between $L_1$ and $L_2$ is called a *simple-triangle* (resp. *triangle* representation of $G$). Simple-triangle and triangle graphs are also known as *PI* and *PI** graphs, respectively [3,5,4,15], where *PI* stands for "Point-Interval"; note that, using this notation, permutation graphs are *PP* (for "Point–Point") graphs, while trapezoid graphs are *II* (for "Interval–Interval") graphs [5]. In particular, both interval and permutation graphs are strictly contained in simple-triangle graphs, which are strictly contained in triangle graphs, which are strictly contained in trapezoid graphs [5,3]. For instance, it is easy to see that every interval graph $G$ is also a simple-triangle graph: given an interval representation of $G$, replace every interval $I_v$ in this representation by an isosceles triangle $T_v$ of unit height, which has the interval $I_v$ as its base. The resulting representation is a simple-triangle representation of $G$, since for any two vertices $u$ and $v$ of $G$, the intervals $I_u$ and $I_v$ intersect if and only if $T_u$ and $T_v$ intersect.

Due to both their interesting structure and their practical applications, trapezoid graphs have attracted many research efforts. In particular, efficient algorithms for several optimization problems that are NP-hard in general graphs have been designed for trapezoid graphs [13,16,12,2,7,24,11], which also apply to triangle and simple-triangle graphs. Furthermore, several efficient algorithms appeared for the recognition problems of both permutation [17,9] and trapezoid graphs [16,20,14]; see [25] for an overview.

In spite of this, the complexity status of both triangle and simple-triangle recognition problems have been the most fundamental open problems on these classes of graphs since their introduction two decades ago [3]. Since, on the one hand, very few subclasses of perfect graphs are known to be NP-hard to recognize (for instance, *perfectly orderable* graphs [22], *EPT* graphs [10], and recently *tolerance* and *bounded tolerance* graphs [21]) and, on the other hand, triangle and simple-triangle graphs lie naturally between permutation and trapezoid graphs, while they share a very similar structure with them, it was plausible that the recognition of triangle and simple-triangle graphs was polynomial.

*Our contribution.* In this article we establish the complexity of recognizing triangle graphs. Namely, we prove that this problem is surprisingly NP-hard, by providing a reduction from the 3SAT problem. Specifically, given a boolean formula formula $\phi$ in conjunctive normal form with three literals in every clause (3-CNF), we construct a trapezoid graph $G_\phi$, which is a triangle graph if and only if $\phi$ is satisfiable. Therefore, as the recognition problems for both triangle and simple-triangle graphs are in the complexity class NP, it follows in particular that the triangle graph recognition problem is NP-complete. This complements the recent surprising result that the recognition of *parallelogram* graphs (i.e. the intersection graphs of parallelograms between two parallel lines $L_1$ and $L_2$), which coincides with *bounded tolerance* graphs, is NP-complete [21].

*Organization of the paper.* Background definitions and properties of trapezoid graphs and their representations are presented in Section 2. In Section 3 we introduce the notion of a *standard trapezoid representation*, the existence of which is a sufficient condition for a trapezoid graph to be a triangle graph. In Sections 4 and 5, we investigate the structure of some specific trapezoid and triangle graphs, respectively, and prove special properties of them. We use these graphs as parts of the gadgets in our reduction of 3SAT to the recognition problem of triangle graphs, which we present in Section 6. Finally, we discuss the presented results and further research in Section 7.

## 2. Triangle and simple-triangle graphs

In this section we provide some notation and properties of trapezoid graphs and their representations, which will be mainly applied in the sequel to triangle and simple-triangle graphs.

**Notation.** We consider in this article simple undirected and directed graphs with no loops or multiple edges. In an undirected graph $G$, the edge between vertices $u$ and $v$ is denoted by $uv$, and in this case $u$ and $v$ are said to be *adjacent* in $G$. Given a graph $G = (V, E)$ and a subset $S \subseteq V$, $G[S]$ denotes the induced subgraph of $G$ on the vertices in $S$. Furthermore, we denote for simplicity by $G - S$ the induced subgraph $G[V \setminus S]$ of $G$. Moreover, given a graph $G$, we denote its vertex set by $V(G)$. A connected graph $G = (V, E)$ is called *k-connected*, where $k \geq 1$, if $k$ is the smallest number of vertices that have to be removed from $G$ such that the resulting graph is disconnected. Furthermore, a vertex $v$ of a 1-connected graph $G$ is called a *cut vertex* of $G$, if $G - \{v\}$ is disconnected. By possibly performing a small shift of the endpoints, we assume throughout the article without loss of generality that all endpoints of the trapezoids (resp. triangles) in a trapezoid (resp. triangle or simple-triangle) representation are distinct [8,11,12]. Given a trapezoid (resp. triangle or simple-triangle) graph $G$ along with a trapezoid (resp. triangle or simple-triangle) representation $R$, we may not distinguish in the following between a

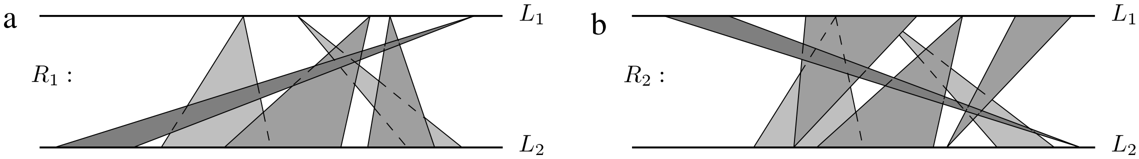


**Fig. 1.** (a) A simple-triangle representation $R_1$ and (b) a triangle representation $R_2$.

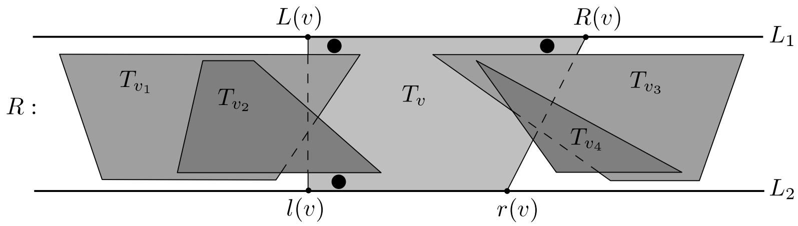


**Fig. 2.** A standard trapezoid representation $R$, in which the trapezoid $T_v$ is left-closed, upper-right-closed, and lower-right-open.

vertex of $G$ and the corresponding trapezoid (resp. triangle) in $R$, whenever it is clear from the context. Moreover, given an induced subgraph $H$ of $G$, we denote by $R[H]$ the restriction of the representation $R$ on the trapezoids (resp. triangles) of $H$.

Consider a trapezoid graph $G = (V, E)$ and a trapezoid representation $R$ of $G$, where for any vertex $u \in V$ the trapezoid corresponding to $u$ in $R$ is denoted by $T_u$. Since trapezoid graphs are also cocomparability graphs (there is a transitive orientation of the complement) [9], we can define the partial order $(V, \ll_R)$, such that $u \ll_R v$, or equivalently $T_u \ll_R T_v$, if and only if $T_u$ lies completely to the left of $T_v$ in $R$ (and thus also $uv \notin E$). Otherwise, if neither $T_u \ll_R T_v$ nor $T_v \ll_R T_u$, we will say that $T_u$ *intersects* $T_v$ in $R$ (and thus also $uv \in E$). Furthermore, we define the total order $<_R$ on the lines $L_1$ and $L_2$ in $R$ as follows. For two points $a$ and $b$ on $L_1$ (resp. on $L_2$), if $a$ lies to the left of $b$ on $L_1$ (resp. on $L_2$), then we will write $a <_R b$.

There are several trapezoid representations of a particular trapezoid graph $G$. For instance, given one such representation $R$, we can obtain another one $R'$ by *vertical axis flipping* of $R$, i.e. $R'$ is the mirror image of $R$ along an imaginary line perpendicular to $L_1$ and $L_2$. Moreover, we can obtain another representation $R''$ of $G$ by *horizontal axis flipping* of $R$, i.e. $R''$ is the mirror image of $R$ along an imaginary line parallel to $L_1$ and $L_2$. We will use extensively these two basic operations throughout the article. For every trapezoid $T_u$ in $R$, where $u \in V$, we define by $l(u)$ and $r(u)$ (resp. $L(u)$ and $R(u)$) the lower (resp. upper) left and right endpoint of $T_u$, respectively (cf. the trapezoid $T_v$ in Fig. 2). Since every triangle and simple-triangle representation is a special type of a trapezoid representation, all the above notions can be also applied to triangle and simple-triangle graphs. Note here that, if $R$ is a simple-triangle representation of $G = (V, E)$, then $L(u) = R(u)$ for every $u \in V$; similarly, if $R$ is a triangle representation of $G$, then $L(u) = R(u)$ or $l(u) = r(u)$ for every $u \in V$. An example of a simple-triangle and a triangle representation is shown in Fig. 1.

It can be easily seen that every triangle (resp. single-triangle) graph $G$ with $n$ vertices has a triangle (resp. single-triangle) representation of $G$, in which the endpoints of the triangles in both lines $L_1$ and $L_2$ are integers between 1 and $2n$. That is, every triangle (resp. single-triangle) graph $G$ with $n$ vertices has a representation with size polynomial on $n$, and thus the recognition problems of both both triangle and simple-triangle graphs are in NP, as the next observation states.

**Observation 1.** *The triangle and simple-triangle graph recognition problems are in the complexity class NP.*

## 3. Standard trapezoid representations

In this section we investigate several properties of trapezoid and triangle graphs and their representations. In particular, we introduce the notion of a *standard trapezoid representation*. We prove that a sufficient condition for a trapezoid graph $G$ to be a triangle graph is that $G$ admits such a standard representation. These properties of trapezoid and triangle graphs, as well as the notion of a standard trapezoid representation will then be used in our reduction for the triangle graph recognition problem. In order to define the notion of a standard trapezoid representation (cf. Definition 3), we first provide the following two definitions regarding an arbitrary trapezoid $T_v$ in a trapezoid representation.

**Definition 1.** Let $R$ be a trapezoid representation of a trapezoid graph $G = (V, E)$ and $T_v$ be a trapezoid in $R$, where $v \in V$. Let $R'$ and $R''$ be the representations obtained by vertical axis flipping and by horizontal axis flipping of $R$, respectively. Then,

- $T_v$ is *upper-right-closed in R* if there exist two vertices $u, w \in N(v)$, such that $T_u \ll_R T_w$, $L(w) <_R R(v)$, and $r(v) <_R l(w)$; otherwise $T_v$ is *upper-right-open in R*,
- $T_v$ is *upper-left-closed in R* if $T_v$ is upper-right-closed in $R'$; otherwise $T_v$ is *upper-left-open in R*,
- $T_v$ is *lower-right-closed in R* if $T_v$ is upper-right-closed in $R''$; otherwise $T_v$ is *lower-right-open in R*,
- $T_v$ is *lower-left-closed in R* if $T_v$ is lower-right-closed in $R'$; otherwise $T_v$ is *lower-left-open in R*.

Intuitively, if the trapezoid $T_v$ is *upper-right-closed* in the trapezoid representation $R$ (cf. Definition 1), then there exists another trapezoid $T_w$ in $R$ that "invades" in $T_v$ only at its upper right corner (cf. the trapezoid $T_{v_3}$ in Fig. 2). In addition, according to Definition 1, there exists another vertex $u \in N(v)$, such that $T_u \ll_R T_w$. Intuitively, the existence of such a trapezoid $T_u$ in $R$ means that, if we move the left endpoints $L(w)$ and $l(w)$ of $T_w$ to the left to cover the whole trapezoid $T_v$, then we will change the graph $G$, since in this case the trapezoid $T_w$ will intersect the trapezoid $T_u$ in the resulting representation.

**Definition 2.** Let $R$ be a trapezoid representation of a trapezoid graph $G = (V, E)$ and $T_v$ be a trapezoid in $R$, where $v \in V$. Then,

- $T_v$ is *right-closed in R* if $T_v$ is both upper-right-closed and lower-right-closed in $R$; otherwise $T_v$ is *right-open in R*,
- $T_v$ is *left-closed in R* if $T_v$ is both upper-left-closed and lower-left-closed in $R$; otherwise $T_v$ is *left-open in R*,
- $T_v$ is *closed* in $R$ if $T_v$ is both right-closed and left-closed in $R$; otherwise $T_v$ is *open* in $R$.

As an example for Definitions 1 and 2, consider the trapezoid representation $R$ in Fig. 2. In this figure, the trapezoid $T_v$ is upper-left-closed and lower-left-closed, as well as upper-right-closed and lower-right-open. Therefore, $T_v$ is left-closed and right-open in $R$, i.e. $T_v$ is open in $R$. For better visibility, we place in Fig. 2 three bold bullets on the upper right, upper left, and lower left endpoints of the trapezoid $T_v$, in order to indicate that $T_v$ is upper-right-closed, upper-left-closed, and lower-left-closed, respectively.

We are now ready to define the notion of a standard trapezoid representation.

**Definition 3.** Let $G = (V, E)$ be a trapezoid graph and $R$ be a trapezoid representation of $G$. If, for every $v \in V$, the trapezoid $T_v$ is open in $R$ or $T_v$ is a triangle in $R$, then $R$ is a *standard trapezoid representation*.

For example, the trapezoid representation $R$ in Fig. 2 is a standard representation. Indeed, none of the trapezoids $T_{v_1}, T_{v_2}, T_{v_3}$ is right-closed or left-closed, while $T_v$ is lower-right-open (and therefore also right-open by Definition 2). Thus, each of the trapezoids $T_v$, $T_{v_1}$, $T_{v_2}$, and $T_{v_3}$ is open in $R$. Moreover, $T_{v_4}$ is a triangle in $R$.

Note that every triangle representation is a standard trapezoid representation by Definition 3. We now provide the main theorem of this section, which states a sufficient condition for a trapezoid graph to be a triangle graph.

**Theorem 1.** *Let $G = (V, E)$ be a trapezoid graph. If there exists a standard trapezoid representation of G, then G is a triangle graph.*

**Proof.** Let $R$ be a standard trapezoid representation of $G$. If $R$ is a triangle representation, then $G$ is clearly a triangle graph. Suppose otherwise that $R$ has a trapezoid $T_v$, where $v \in V$, that is not a triangle in $R$. We will construct a triangle representation $R^*$ of $G$. Since $R$ is standard by assumption, $T_v$ is right-open or left-open in $R$ by Definition 2. By possibly performing a vertical axis flipping, we may assume without loss of generality that $T_v$ is right-open in $R$. That is, $T_v$ is upper-right-open or lower-right-open in $R$ by Definition 1. Similarly, by possibly performing a horizontal axis flipping, we may assume without loss of generality that $T_v$ is upper-right-open in $R$.

We construct now from $R$ a new trapezoid representation of $G$, as follows. First, for every vertex $w \in V$ with $L(v) <_R L(w) <_R R(v)$ and $r(v) <_R l(w)$, we move the upper left endpoint $L(w)$ of $T_w$ directly before $L(v)$ on the line $L_1$. Note that $w \in N(v)$ for every such vertex $w$. Moreover, in the case where the upper left endpoint $L(w)$ of $T_w$ coincides with its upper right endpoint $R(w)$ in $R$, i.e. if $T_w$ is a triangle in $R$ with one point on $L_1$, we also move the upper right endpoint $R(w)$ of $T_w$ to the same position as its upper left endpoint $L(w)$ in $R'$. That is, if $T_w$ is a triangle in $R$, it remains a triangle also in $R'$. During the movement of all these endpoints, we keep the same relative positions among them on $L_1$ as in the initial trapezoid representation $R$. Then, we reduce the trapezoid $T_v$ to a triangle, by moving the upper right endpoint $R(v)$ of $T_v$ to the left until it coincides with its upper left endpoint $L(v)$. Let $R'$ be the resulting trapezoid representation. An example of the construction of $R'$ is illustrated in Fig. 3.

We will prove that $R'$ is a representation of the same graph $G$. First recall that, during the transformation of $R$ to $R'$, we moved the endpoints $L(w)$ of the trapezoids $T_w$ for every vertex $w$, for which $L(v) <_R L(w) <_R R(v)$ and $r(v) <_R l(w)$ (cf. $w_1$, $w_2$, and $w_3$ in Fig. 3). Suppose such a trapezoid $T_w$ intersects a new trapezoid $T_u$ in $R'$, while $T_w$ did not intersect $T_u$ in $R$. That is, $T_u \ll_R T_w$. Then, since $L(w)$ came directly before $L(v)$ on the line $L_1$, it follows that $L(v) <_R R(u) <_R L(w) <_R R(v)$, and thus $T_u$ intersects $T_v$ in $R$, i.e. $u \in N(v)$. That is, there exist two vertices $u, w \in N(v)$, such that $T_u \ll_R T_w$, $L(w) <_R R(v)$, and $r(v) <_R l(w)$, and thus $T_v$ is upper-right-closed in $R$ by Definition 1, which is a contradiction to the assumption. Therefore, $T_w$ does not intersect any new trapezoid in $R'$.

Let $L(w) \neq R(w)$ in $R$, i.e. $T_w$ is not a triangle in $R$ with one point on $L_1$ (cf. $w_3$ in Fig. 3). In this case, the upper right endpoint $R(w)$ of $T_w$ remains the same in both $R$ and $R'$, and thus $T_w$ increases during the transformation of $R$ to $R'$. Therefore, if $L(w) \neq R(w)$ in $R$, then $T_w$ keeps in $R'$ all its intersections with other trapezoids.

Let $L(w) = R(w)$ in $R$, i.e. $T_w$ is a triangle in $R$ with one point on $L_1$ (cf. $w_1$ and $w_2$ in Fig. 3). Recall that in this case, we also move during the transformation of $R$ to $R'$ the upper right endpoint $R(w)$ of $T_w$ to the same position as its upper left endpoint $L(w)$ in $R'$. Suppose that, after this movement, $T_w$ misses in $R'$ its intersection with a trapezoid $T_x$ in $R$. That is, $T_w$ intersects $T_x$ in $R$, while $T_w \ll_{R'} T_x$. Therefore, $L(x) <_R L(w) = R(w) <_R R(v)$ and $r(v) <_R l(w) \leq_R r(w) <_R l(x)$, i.e. $L(x) <_R R(v)$ and $r(v) <_R l(x)$. We distinguish now the two cases regarding the relative position of the endpoints $L(v)$ and $L(x)$ in the initial representation $R$. Let first $L(v) <_R L(x)$. In this case, $L(v) <_R L(x) <_R R(v)$ and $r(v) <_R l(x)$. Therefore, the endpoint $L(x)$ of

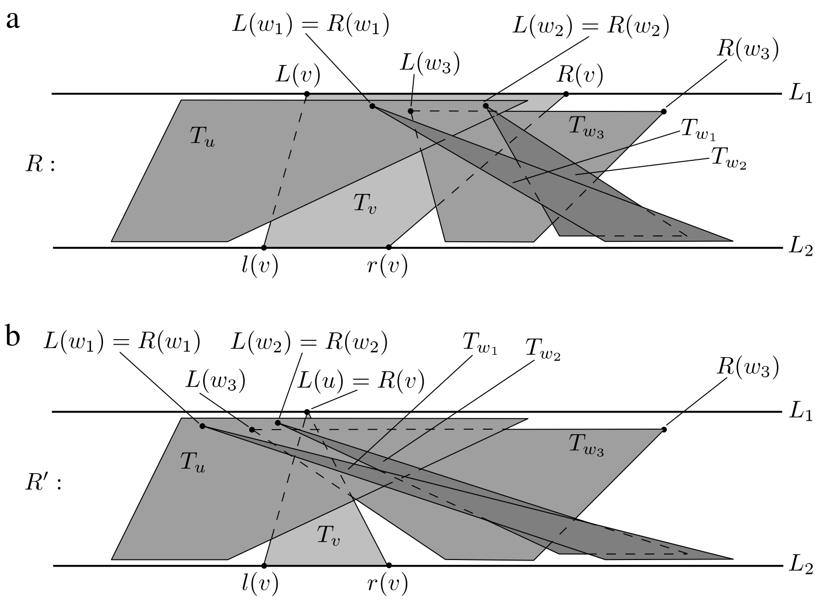


**Fig. 3.** (a) A standard trapezoid representation $R$ of a trapezoid graph $G$ and (b) the transformation of $R$ to a trapezoid representation $R'$ of $G$ with one triangle more.

$T_x$ is moved directly before $L(v)$ on the line $L_1$, while the relative position of $L(x)$ and $L(w)$ remains the same in both $R$ and $R'$. That is, $L(x) <_{R'} L(w)$, which is a contradiction, since $T_w \ll_{R'} T_x$ by assumption. Let now $L(x) <_R L(v)$. Then, $L(x)$ remains the same in both $R$ and $R'$ , while $L(w)$ is moved directly before $L(v)$ in $R'$. That is, $L(x) <_{R'} L(w) <_{R'} L(v)$, which is again a contradiction, since $T_w \ll_{R'} T_x$ by assumption. Therefore, if $L(w) = R(w)$ in $R$, then $T_w$ keeps in $R'$ all its intersections with other trapezoids.

Recall now that we reduced the trapezoid $T_v$ to a triangle (cf. Fig. 3(b)). Suppose that, after this operation, $T_v$ misses in $R'$ its intersection with a trapezoid $T_x$. That is, $T_v$ intersects a trapezoid $T_x$ in $R$, while $T_v$ does not intersect $T_x$ in $R'$, i.e. $T_v \ll_{R'} T_x$. Therefore, $L(v) <_R L(x) <_R R(v)$ and $r(v) <_R l(x)$. Thus, the upper left endpoint $L(x)$ is moved directly before $L(v)$ on the line $L_1$ during the transformation of $R$ to $R'$, which is a contradiction, since $T_v \ll_{R'} T_x$. Therefore, $T_v$ keeps all its intersections in $R'$. Thus, $R'$ is a trapezoid representation of the same graph $G$, in which $T_v$ is a triangle, while all triangles in $R$ remain also triangles n $R'$ (cf. $w_1$ and $w_2$ in Fig. 3).

After applying iteratively the above construction for every trapezoid $T_v$ that is not a triangle in $R$, we obtain a triangle representation $R^*$ of $G$, i.e. $G$ is a triangle graph. This completes the proof of the theorem. □

## 4. Basic constructions of trapezoid graphs

In this section we investigate some small trapezoid graphs and prove special properties of them. These graphs will then be used as parts of the gadgets in our reduction of 3SAT to the recognition problem of triangle graphs in Section 6. For simplicity of the presentation, we do not distinguish in the sequel of the article between a vertex $v$ of a trapezoid graph $G$ and the trapezoid $T_v$ of $v$ in a trapezoid representation of $G$.

**Lemma 1.** *Let $G = (V, E)$ be the trapezoid graph induced by the trapezoid representation of Fig. 4(a). Then, in* any *trapezoid representation $R$ of $G$, such that $v \ll_R v'$,*

- *$v$ is upper-right-closed in $R$ and $v'$ is lower-left-closed in $R$, or*
- *$v$ is lower-right-closed in $R$ and $v'$ is upper-left-closed in $R$.*

**Proof.** Consider a trapezoid representation $R$ of $G$, such that $v \ll_R v'$. Since the vertices $v_1$ and $v_2$ are indistinguishable, as $N(v_1) = N(v_2)$, we may assume without loss of generality that $v_1 \ll_R v_2$. Furthermore, note that if both $R(v) <_R L(v_2)$ and $r(v) <_R l(v_2)$, then $v \ll_R v_2$, which is a contradiction, since $v_2$ intersects $v$ in $R$. Therefore, $L(v_2) <_R R(v)$ or $l(v_2) <_R r(v)$.

Suppose that $L(v_2) <_R R(v)$. An example of such a trapezoid representation $R$ is the representation $R_1$ in Fig. 4(a). Then, since $v_1 \ll_R v_2$ and $v \ll_R v'$ by assumption, it follows that $R(v_1) <_R L(v_2) <_R R(v) <_R L(v')$, i.e. $R(v_1) <_R L(v')$. Now, if $r(v_1) <_R l(v')$, then $v_1 \ll_R v'$, which is a contradiction, since $v_1$ intersects $v'$ in $R$. Thus $l(v') <_R r(v_1)$. Therefore, since $v \ll_R v'$ and $v_1 \ll_R v_2$ by assumption, it follows that $r(v) <_R l(v') <_R r(v_1) <_R l(v_2)$, i.e. $r(v) <_R l(v_2)$. Summarizing, there exist two vertices $v_1, v_2 \in N(v)$, such that $v_1 \ll_R v_2$, $L(v_2) <_R R(v)$, and $r(v) <_R l(v_2)$, and thus $v$ is upper-right-closed in $R$ by Definition 1. Moreover, $R(v_1) <_R L(v')$ and $l(v') <_R r(v_1)$, and thus $v'$ is lower-left-closed in $R$ by Definition 1.

Suppose now that $l(v_2) <_R r(v)$. Consider the trapezoid representation $R'$ of $G$ that is obtained by performing a horizontal axis flipping of $R$. Examples of these trapezoid representations $R$ and $R'$ are the representations $R_2$ and $R_1$ in Fig. 4(b) and (a),

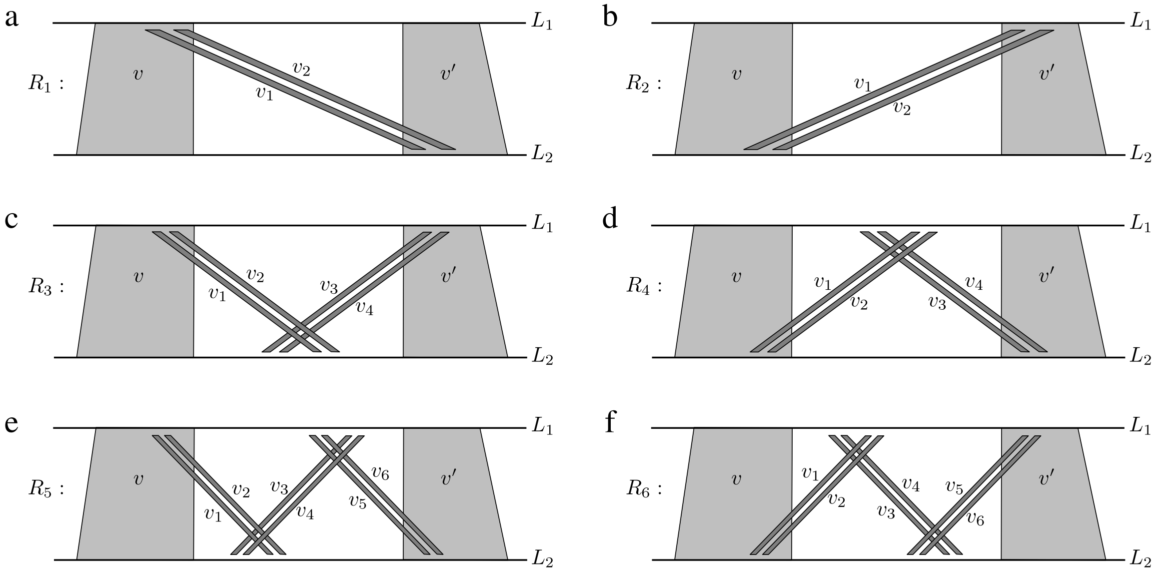


**Fig. 4.** Six basic trapezoid representations.

respectively. Note that $L(v_2) <_{R'} R(v)$, since $l(v_2) <_R r(v)$. Moreover, it remains $v_1 \ll_{R'} v_2$, since also $v_1 \ll_R v_2$. Therefore, it follows by the previous paragraph that $v$ is upper-right-closed in $R'$ and that $v'$ is lower-left-closed in $R'$. Thus, $v$ is lower-right-closed in $R$ and $v'$ is upper-left-closed in $R$. This completes the proof of the lemma. □

The next two lemmas concern similar properties of the graphs induced by the trapezoid representations of Fig. 4(c) and (e), respectively.

**Lemma 2.** *Let $G = (V, E)$ be the trapezoid graph induced by the trapezoid representation of Fig. 4(c). Then, in* any *trapezoid representation $R$ of $G$, such that $v \ll_R v'$,*

- *$v$ is upper-right-closed in $R$ and $v'$ is upper-left-closed in $R$, or*
- *$v$ is lower-right-closed in $R$ and $v'$ is lower-left-closed in $R$.*

**Proof.** Consider a trapezoid representation $R$ of $G$, such that $v \ll_R v'$. Since the vertices $v_3$ and $v_4$ are indistinguishable, as $N(v_3) = N(v_4)$, we may assume without loss of generality that $v_3 \ll_R v_4$. Furthermore, since $v \ll_R v'$ and $v_1 \in N(v) \setminus N(v')$, it follows that $v_1 \ll_R v'$. Moreover, similarly to the proof of Lemma 1, note that if both $R(v) <_R L(v_2)$ and $r(v) <_R l(v_2)$, then $v \ll_R v_2$, which is a contradiction, since $v_2$ intersects $v$ in $R$. Therefore, $L(v_2) <_R R(v)$ or $l(v_2) <_R r(v)$.

Suppose that $L(v_2) <_R R(v)$. An example of such a trapezoid representation $R$ is the representation $R_3$ in Fig. 4(c). Consider the induced subgraph $G_1 = G[\{v, v_1, v_2, v_4\}]$ of $G$; note that $G_1$ is the graph investigated in Lemma 1, where vertex $v_4$ corresponds to vertex $v'$ of Lemma 1. Similarly to the proof of Lemma 1 in the corresponding case, it follows that $v$ is upper-right-closed in $R$ and that $l(v_4) <_R r(v_1)$. Therefore, since $v_3 \ll_R v_4$ and $v_1 \ll_R v'$, it follows that $r(v_3) <_R l(v_4) <_R r(v_1) <_R l(v')$, i.e. $r(v_3) <_R l(v')$. Now, if $R(v_3) <_R L(v')$, then $v_3 \ll_R v'$, which is a contradiction, since $v_3$ intersects $v'$ in $R$. Thus $L(v') <_R R(v_3)$. Summarizing, there exist two vertices $v_3, v_4 \in N(v')$, such that $v_3 \ll_R v_4$, $r(v_3) <_R l(v')$, and $L(v') <_R R(v_3)$, and thus $v'$ is upper-left-closed in $R$ by Definition 1.

Suppose now that $l(v_2) <_R r(v)$. Consider the trapezoid representation $R'$ of $G$ that is obtained by performing a horizontal axis flipping of $R$. Examples of these trapezoid representations $R$ and $R'$ are the representations $R_4$ and $R_3$ in Fig. 4(d) and (c), respectively. Note that $L(v_2) <_{R'} R(v)$, since $l(v_2) <_R r(v)$. Moreover, it remains $v_3 \ll_{R'} v_4$, since also $v_3 \ll_R v_4$. Therefore, it follows by the previous paragraph that $v$ is upper-right-closed in $R'$ and that $v'$ is upper-left-closed in $R'$. Thus, $v$ is lower-right-closed in $R$ and $v'$ is lower-left-closed in $R$. This completes the proof of the lemma. □

**Lemma 3.** *Let $G = (V, E)$ be the trapezoid graph induced by the trapezoid representation of Fig. 4(e). Then, in* any *trapezoid representation $R$ of $G$, such that $v \ll_R v'$,*

- *$v$ is upper-right-closed in $R$ and $v'$ is lower-left-closed in $R$, or*
- *$v$ is lower-right-closed in $R$ and $v'$ is upper-left-closed in $R$.*

**Proof.** Consider a trapezoid representation $R$ of $G$, such that $v \ll_R v'$. Since the vertices $v_5$ and $v_6$ are indistinguishable, as $N(v_5) = N(v_6)$, we may assume without loss of generality that $v_5 \ll_R v_6$. Furthermore, since $v \ll_R v'$ and $v_1 \in N(v) \setminus N(v')$, it follows that $v_1 \ll_R v'$. Therefore, since $v_1 \ll_R v'$ and $v_3 \in N(v_1) \setminus N(v')$, it follows that $v_3 \ll_R v'$. Moreover, similarly to the proof of Lemma 1, note that if both $R(v) <_R L(v_2)$ and $r(v) <_R l(v_2)$, then $v \ll_R v_2$, which is a contradiction, since $v_2$ intersects $v$ in $R$. Therefore, $L(v_2) <_R R(v)$ or $l(v_2) <_R r(v)$.

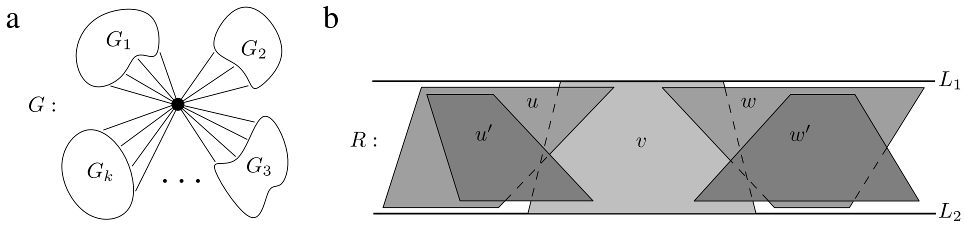


**Fig. 5.** (a) A 1-connected graph $G$, where $G - \{v\}$ has $k \geq 2$ connected components $G_1, G_2, \ldots, G_k$ and (b) a trapezoid representation for such a graph $G$.

Suppose that $L(v_2) <_R R(v)$. An example of such a trapezoid representation $R$ is the representation $R_5$ in Fig. 4(e). Consider the induced subgraph $G_1 = G[\{v, v_1, v_2, v_3, v_4, v_6\}]$ of $G$; note that $G_1$ is the graph investigated in Lemma 2, where vertex $v_6$ corresponds to vertex $v'$ of Lemma 2. Similarly to the proof of Lemma 2 in the corresponding case, it follows that $v$ is upper-right-closed in $R$ and that $L(v_6) <_R R(v_3)$. Therefore, since $v_5 \ll_R v_6$ and $v_3 \ll_R v'$, it follows that $R(v_5) <_R L(v_6) <_R R(v_3) <_R L(v')$, i.e. $R(v_5) <_R L(v')$. Now, if $r(v_5) <_R l(v')$, then $v_5 \ll_R v'$, which is a contradiction, since $v_5$ intersects $v'$ in $R$. Thus $l(v') <_R r(v_5)$. Summarizing, there exist two vertices $v_5, v_6 \in N(v')$, such that $v_5 \ll_R v_6$, $R(v_5) <_R L(v')$, and $l(v') <_R r(v_5)$, and thus $v'$ is lower-left-closed in $R$ by Definition 1.

Suppose now that $l(v_2) <_R r(v)$. Consider the trapezoid representation $R'$ of $G$ that is obtained by performing a horizontal axis flipping of $R$. Examples of these trapezoid representations $R$ and $R'$ are the representations $R_6$ and $R_5$ in Fig. 4(f) and (e), respectively. Note that $L(v_2) <_{R'} R(v)$, since $l(v_2) <_R r(v)$. Moreover, it remains that $v_5 \ll_{R'} v_6$, since also $v_5 \ll_R v_6$. Therefore, it follows by the previous paragraph that $v$ is upper-right-closed in $R'$ and that $v'$ is lower-left-closed in $R'$. Thus, $v$ is lower-right-closed in $R$ and $v'$ is upper-left-closed in $R$. This completes the proof of the lemma. □

## 5. Basic constructions of triangle graphs

In this section we investigate the structure of some specific triangle graphs and devise special properties of them. As triangle graphs are also trapezoid graphs, in order to prove these properties, we use some of the results provided in Section 4. Similarly to the trapezoid graphs investigated in Section 4, also the investigated graphs of the present section will then be used as gadgets in our reduction for the triangle graph recognition problem in Section 6. Before investigating any specific triangle graph, we first provide in the next theorem a generic result that concerns the triangle representations of the 1-connected triangle graphs. An example of a 1-connected graph $G$ with a cut vertex $v$ is illustrated in Fig. 5(a).

**Theorem 2.** *Let $G = (V, E)$ be a 1-connected triangle graph and $v \in V$ be a cut vertex of $G$. Then, in* any *triangle representation $R$ of $G$, the trapezoid of $v$ is open in $R$.*

**Proof.** Let $R$ be any triangle representation of $G$. For the sake of contradiction, suppose that $v$ is closed in $R$, i.e. $v$ is both left-closed and right-closed in $R$. We will prove that, in this case, the trapezoid $T_v$ has four distinct endpoints, and thus $T_v$ is not a triangle in $R$, which comes in contradiction to the assumption that $R$ is a triangle representation. Let $G_1, G_2, \ldots, G_k$ be the connected components of $G - \{v\}$, where $k \geq 2$. Then, for any $i \neq j$, either all trapezoids of $G_i$ lie completely to the left or to the right of all trapezoids of $G_j$ in $R$.

Note that $v$ is upper-right-closed in $R$ (as $v$ is right-closed in $R$ by assumption). Therefore, in particular, there exists by Definition 1 a vertex $w \in N(v)$, such that $L(w) <_R R(v)$ and $r(v) <_R l(w)$ (cf. Fig. 5(b)). Since $v$ is also lower-right-closed in $R$, there exists a vertex $w' \in N(v)$, such that $l(w') <_R r(v)$ and $R(v) <_R L(w')$ (cf. Fig. 5(b)). Summarizing, $l(w') <_R r(v) <_R l(w)$ and $L(w) <_R R(v) <_R L(w')$. Therefore, in particular, the trapezoids of $w$ and of $w'$ intersect in $R$, i.e. $ww' \in E$. Therefore, both $w, w' \in V(G_i)$, for some $i = 1, 2, \ldots, k$.

Similarly, since $v$ is upper-left-closed and lower-left-closed in $R$ (as $v$ is left-closed in $R$ by assumption), there exist two vertices $u, u' \in N(v)$, such that $R(u') <_R L(v) <_R R(u)$ and $r(u) <_R l(v) <_R r(u')$, cf. Fig. 5(b). Therefore, in particular, the trapezoids of $u$ and of $u'$ intersect in $R$, i.e. $uu' \in E$. Thus, both $u, u' \in V(G_j)$, for some $j = 1, 2, \ldots, k$.

Suppose that $j = i$, i.e. the vertices $w, w', u, u'$ belong to the same connected component of $G - \{v\}$. Consider now another connected component $G_\ell$ of $G - \{v\}$, where $\ell \neq i$. Note that $G_\ell$ exists, since $G - \{v\}$ has at least two connected components. Recall that either all trapezoids of $G_\ell$ lie to the left or to the right of all trapezoids of $G_i$ in $R$. Suppose that the trapezoids of $G_\ell$ lie to the left of the trapezoids of $G_i$ in $R$, i.e. $x \ll_R y$ for every $x \in V(G_\ell)$ and $y \in V(G_i)$. Then, since $r(u) <_R l(v)$ and $R(u') <_R L(v)$, and since $u, u' \in V(G_i)$, it follows that $x \ll_R v$ for every $x \in V(G_\ell)$. Thus no vertex of $G_\ell$ is adjacent to $v$, i.e. $G$ is not connected, which is a contradiction by the assumption on $G$. Similarly, suppose that the trapezoids of $G_\ell$ lie to the right of the trapezoids of $G_i$ in $R$, i.e. $y \ll_R x$ for every $x \in V(G_\ell)$ and $y \in V(G_i)$. Then, since $r(v) <_R l(w)$ and $R(v) <_R L(w')$, and since $w, w' \in V(G_i)$, it follows that $v \ll_R x$ for every $x \in V(G_\ell)$. Thus no vertex of $G_\ell$ is adjacent to $v$, i.e. $G$ is not connected, which is again a contradiction. Therefore $j \neq i$.

Recall that $r(u) <_R l(v) \leq_R r(v) <_R l(w)$, and thus $r(u) <_R l(w)$. Furthermore, recall that $R(u') <_R L(v) \leq_R R(v) <_R L(w')$, and thus $R(u') <_R L(w')$. Moreover, since $u, u' \in V(G_j)$ and $w, w' \in V(G_i)$, where $j \neq i$, it follows that $uw, u'w' \notin E$, and thus $u \ll_R w$ and $u' \ll_R w'$. Summarizing, there exist four distinct vertices $u, u', w, w' \in N(v)$, such that $L(v) <_R R(u) <_R L(w) <_R R(v)$ and $l(v) <_R r(u') <_R l(w') <_R r(v)$. Therefore $L(v) <_R R(v)$ and $l(v) <_R r(v)$, i.e. $L(v) \neq R(v)$ and $l(v) \neq r(v)$, which contradicts the fact that $R$ is a triangle representation. An example of such a forbidden representation $R$, where $V(G_1) = \{u, u'\}$ and $V(G_2) = \{w, w'\}$, is illustrated in Fig. 5(b). Therefore, $v$ is open in $R$. $\square$

We now use the generic Theorem 2, as well as the results of Section 4, in order to prove some properties of the trapezoid representations of Fig. 6. Note that, although the representations of Fig. 6 are not triangle representations, they are standard trapezoid representations, and thus the graphs induced by these representations are triangle graphs by Theorem 1.

**Lemma 4.** *Let $G = (V, E)$ be the triangle graph induced by the trapezoid representation of Fig. 6(a). Then, in* any *triangle representation $R$ of $G$, such that $a_7 \ll_R u$, $u$ is left-open in $R$ if and only if $w$ is right-open in $R$.*

**Proof.** Let $R$ be a triangle representation of $G$, such that $a_7 \ll_R u$. Note that $G - \{u, w\}$ has the two connected components $G_1 = G[a_1, a_2, a_3, a_4, a_5, a_6, a_7]$ and $G_2 = G[v, b_1, b_2, b_3, b_4, b_5, b_6]$, and thus one of these two induced subgraphs of $G$ lies completely to the left of the other in $R$. If $v \ll_R a_7 \ll_R u$, then $a_7$ would intersect with a triangle of $G_2$, which is a contradiction, since $a_7 \in V(G_1)$. Furthermore, if $a_7 \ll_R v \ll_R u$, then $v$ would intersect with a triangle of $G_1$, which is a contradiction, since $v \in V(G_2)$. Therefore $a_7 \ll_R u \ll_R v$; similarly, $a_7 \ll_R w \ll_R v$. Therefore, every triangle of $G_1$ must lie completely to the left of every triangle of $G_2$ in $R$.

($\Rightarrow$) Suppose that $u$ is left-open in $R$, i.e. $u$ is upper-left-open or lower-left-open in $R$. By possibly performing a horizontal axis flipping of $R$, we may assume without loss of generality that $u$ is lower-left-open in $R$. Consider the induced subgraphs $H_1 = G[\{a_7, a_1, a_2, u\}]$ and $H_2 = G[\{a_7, a_1, a_2, w\}]$ of $G$. Note that both $H_1$ and $H_2$ are isomorphic to the graph investigated in Lemma 1. Since $u$ is assumed to be lower-left-open in $R$ (and thus also in the restriction $R[H_1]$ of the triangle representation $R$), Lemma 1 implies that $u$ is upper-left-closed and $a_7$ is lower-right-closed in $R[H_1]$. Therefore, $a_7$ is lower-right-closed also in the restriction $R[H_1 - \{u\}] = R[H_2 - \{w\}]$ of $R$. Thus, Lemma 1 implies that $a_7$ is lower-right-closed and $w$ is upper-left-closed in the restriction $R[H_2]$ of $R$, and thus $w$ is upper-left-closed in $R$.

Consider now the induced subgraphs $H_3 = G[\{a_7, a_3, a_4, u\}]$ and $H_4 = G[\{a_7, a_3, a_4, a_5, a_6, w\}]$ of $G$. Note that $H_3$ is isomorphic to the graph investigated in Lemma 1, while $H_4$ is isomorphic to the graph investigated in Lemma 2. Since $u$ is assumed to be lower-left-open in $R$ (and thus also in $R[H_3]$), Lemma 1 implies that $u$ is upper-left-closed and $a_7$ is lower-right-closed in $R[H_3]$. Therefore, $a_7$ is lower-right-closed also in the restriction $R[H_3 - \{u\}] = R[H_4 - \{a_5, a_6, w\}]$ of the triangle representation $R$. Thus, Lemma 2 implies that $a_7$ is lower-right-closed and $w$ is lower-left-closed in the restriction $R[H_4]$ of $R$, and thus $w$ is lower-left-closed in $R$. Therefore, since $w$ is also upper-left-closed in $R$ by the previous paragraph, it follows that $w$ is left-closed in $R$.

Recall that $R$ is a triangle representation by assumption, and thus the restriction $R[G - \{u\}]$ is also a triangle representation. Moreover, since $w$ is left-closed in $R$, it follows that $w$ is also left-closed in $R[G - \{u\}]$. Note now that the connected graph $G - \{u\}$ satisfies the conditions of Theorem 2. Indeed, $w$ is a cut vertex of $G - \{u\}$ and $(G - \{u\}) - \{w\}$ has the two connected components $G_1 = G[a_1, a_2, a_3, a_4, a_5, a_6, a_7]$ and $G_2 = G[v, b_1, b_2, b_3, b_4, b_5, b_6]$. Therefore, since $w$ is left-closed in $R[G - \{u\}]$, Theorem 2 implies that $w$ is right-open in $R[G - \{u\}]$, and thus also $w$ is right-open in $R$.

($\Leftarrow$) Consider the triangle representation $R'$ of $G$ that is obtained by performing a vertical axis flipping of $R$. Note that $v \ll_{R'} w$, since $w \ll_R v$. Furthermore, note that there is a trivial automorphism of $G$, which maps vertex $u$ to $w$, vertex $a_7$ to $v$, and the vertices $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ to the vertices $\{b_1, b_2, b_3, b_4, b_5, b_6\}$, in this order. That is, the relation $a_7 \ll_R u$ in the representation $R$ is mapped by this automorphism to the relation $v \ll_{R'} w$ in the representation $R'$. It follows now directly by the necessity part ($\Rightarrow$) that, if $w$ is left-open in $R'$, then $u$ is right-open in $R'$. That is, if $w$ is right-open in $R$, then $u$ is left-open in $R$. $\square$

Now, using Lemma 4, we can prove the next two lemmas.

**Lemma 5.** *Let $G = (V, E)$ be the triangle graph induced by the trapezoid representation of Fig. 6(a). Then, in* any *triangle representation $R$ of $G$, such that $a_7 \ll_R u$, $u$ is left-open in $R$ if and only if $v$ is left-open in $R$.*

**Proof.** Let $R$ be a triangle representation of $G$, such that $a_7 \ll_R u$. Recall by the proof of Lemma 4 that $w \ll_R v$.

($\Rightarrow$) Suppose that $u$ is left-open in $R$. Then, $w$ is right-open in $R$ by Lemma 4, i.e. $w$ is upper-right-open or lower-right-open in $R$. By possibly performing a horizontal axis flipping of $R$, we may assume without loss of generality that $w$ is upper-right-open in $R$. Consider the induced subgraphs $H_1 = G[\{w, b_1, b_2, v\}]$ and $H_2 = G[\{w, b_5, b_6, v\}]$ of $G$. Note that both $H_1$ and $H_2$ are isomorphic to the graph investigated in Lemma 1. Since $w$ is assumed to be upper-right-open in $R$ (and thus also in both restrictions $R[H_1]$ and $R[H_2]$ of $R$), Lemma 1 implies that $w$ is lower-right-closed and $v$ is upper-left-closed in both $R[H_1]$ and $R[H_2]$, and thus $v$ is upper-left-closed in $R$. Therefore, since $b_1, b_2, b_5, b_6$ are the only neighbors of $v$ in $G$, it follows that $v$ is lower-left-open in $R$, and thus $v$ is left-open in $R$.

($\Leftarrow$) Suppose that $v$ is left-open in $R$, i.e. $v$ is upper-left-open or lower-left-open in $R$. By possibly performing a horizontal axis flipping of $R$, we may assume without loss of generality that $v$ is lower-left-open in $R$. Consider the induced subgraphs $H_3 = G[\{u, b_1, b_2, v\}]$ and $H_4 = G[\{u, b_3, b_4, b_5, b_6, v\}]$ of $G$. Note that $H_3$ is isomorphic to the graph investigated in Lemma 1, while $H_4$ is isomorphic to the graph investigated in Lemma 2. Since $v$ is assumed to be lower-left-open in $R$, it follows that $v$ is lower-left-open also in the restrictions $R[H_3]$ and $R[H_4]$ of $R$. Therefore, Lemma 1 implies that $v$ is upper-left-closed and $u$ is lower-right-closed in $R[H_3]$, and thus also in $R$. Similarly, Lemma 2 implies that $v$ is upper-left-closed and $u$ is upper-right-closed in $R[H_4]$, and thus also in $R$. Summarizing, $u$ is both lower-right-closed and upper-right-closed in $R$, and thus $u$ is right-closed in $R$.

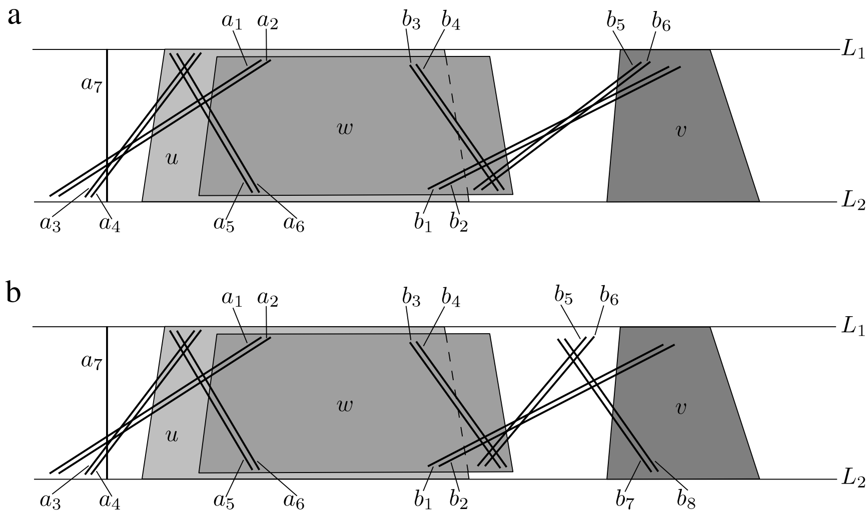


**Fig. 6.** Two basic trapezoid representations.

Recall that $R$ is a triangle representation by assumption, and thus the restriction $R[G - \{w\}]$ is also a triangle representation. Moreover, since $u$ is right-closed in $R$, it follows that $u$ is also right-closed in $R[G - \{w\}]$. Note now that the connected graph $G - \{w\}$ satisfies the conditions of Theorem 2. Indeed, $u$ is a cut vertex of $G - \{w\}$ and $(G - \{w\}) - \{u\}$ has the two connected components $G_1 = G[a_1, a_2, a_3, a_4, a_5, a_6, a_7]$ and $G_2 = G[v, b_1, b_2, b_3, b_4, b_5, b_6]$. Therefore, since $u$ is right-closed in $R[G - \{w\}]$, Theorem 2 implies that $u$ is left-open in $R[G - \{w\}]$, and thus also $u$ is left-open in $R$. □

**Lemma 6.** *Let $G = (V, E)$ be the triangle graph induced by the trapezoid representation of Fig. 6(b). Then, in* any *triangle representation $R$ of $G$, such that $a_7 \ll_R u$, $u$ is left-open in $R$ if and only if $v$ is left-closed in $R$.*

**Proof.** Let $R$ be a triangle representation of $G$, such that $a_7 \ll_R u$. Note that the induced subgraph $H = G - \{b_8, v\}$ is isomorphic to the graph investigated in Lemmas 4 and 5. That is, $H$ is isomorphic to the graph induced by the trapezoid representation of Fig. 6(a).

($\Rightarrow$) Suppose that $u$ is left-open in $R$. Then, $u$ is also left-open in the restriction $R[H]$ of $R$. Therefore, $w$ is right-open in $R[H]$ by Lemma 4, and thus $w$ is also right-open in $R$. That is, $w$ is upper-right-open or lower-right-open in $R$. By possibly performing a horizontal axis flipping of $R$, we may assume without loss of generality that $w$ is upper-right-open in $R$. Consider the induced subgraphs $H_1 = G[\{w, b_1, b_2, v\}]$ and $H_2 = G[\{w, b_5, b_6, b_7, b_8, v\}]$ of $G$. Note that $H_1$ is isomorphic to the graph investigated in Lemma 1, while $H_2$ is isomorphic to the graph investigated in Lemma 2. Since $w$ is assumed to be upper-right-open in $R$, it follows that $w$ is upper-right-open also in the restrictions $R[H_1]$ and $R[H_2]$ of $R$. Therefore, Lemma 1 implies that $w$ is lower-right-closed and $v$ is upper-left-closed in $R[H_1]$, and thus also in $R$. Similarly, Lemma 2 implies that $w$ is lower-right-closed and $v$ is lower-left-closed in $R[H_2]$, and thus also in $R$. Summarizing, $v$ is both upper-left-closed and lower-left-closed in $R$, and thus $v$ is left-closed in $R$.

($\Leftarrow$) Suppose that $u$ is left-closed in $R$. Recall that $R$ is a triangle representation by assumption, and thus the restriction $R[G - \{w\}]$ is also a triangle representation. Moreover, since $u$ is left-closed in $R$, it follows that $u$ is also left-closed in $R[G - \{w\}]$. Note now that the connected graph $G - \{w\}$ satisfies the conditions of Theorem 2. Indeed, $u$ is a cut vertex of $G - \{w\}$ and $(G - \{w\}) - \{u\}$ has the two connected components $G_1 = G[a_1, a_2, a_3, a_4, a_5, a_6, a_7]$ and $G_2 = G[v, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8]$. Therefore, since $u$ is left-closed in $R[G - \{w\}]$, Theorem 2 implies that $u$ is right-open in $R[G - \{w\}]$, and thus also $u$ is right-open in $R$. That is, $u$ is upper-right-open or lower-right-open in $R$. By possibly performing a horizontal axis flipping of $R$, we may assume without loss of generality that $u$ is upper-right-open in $R$.

Consider the induced subgraphs $H_3 = G[\{u, b_1, b_2, v\}]$ and $H_4 = G[\{u, b_3, b_4, b_5, b_6, b_7, b_8, v\}]$ of $G$. Note that $H_3$ is isomorphic to the graph investigated in Lemma 1, while $H_4$ is isomorphic to the graph investigated in Lemma 3. Since $u$ is assumed to be upper-right-open in $R$, it follows that $u$ is upper-right-open also in the restrictions $R[H_3]$ and $R[H_4]$ of $R$. Therefore, Lemma 1 implies that $u$ is lower-right-closed and $v$ is upper-left-closed in $R[H_3]$, and thus also in $R$. Similarly, Lemma 3 implies that $u$ is lower-right-closed and $v$ is upper-left-closed in $R[H_4]$. That is, $v$ is upper-left-closed in both $R[H_3]$ and $R[H_4]$, and thus $v$ is upper-left-closed in $R$. Therefore, since $b_1, b_2, b_7, b_8$ are the only neighbors of $v$ in $G$, it follows that $v$ is lower-left-open in $R$, and thus $v$ is left-open in $R$. This completes the proof of the lemma. □

## 6. The recognition of triangle graphs

In this section we provide a reduction from the *three-satisfiability (3SAT)* problem to the problem of recognizing whether a given graph is a triangle graph. Given a boolean formula $\phi$ in conjunctive normal form with three literals in each clause (3-CNF), $\phi$ is *satisfiable* if there is a truth assignment of $\phi$, such that every clause contains at least one true literal. The

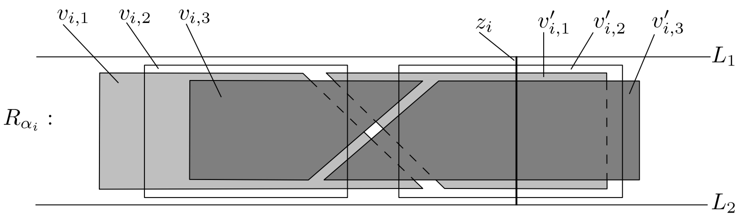


**Fig. 7.** The construction $R_{\alpha_i}$ that is associated to the clause $\alpha_i$ of the formula $\phi$, for $i \in \{1, 2, \ldots, k\}$.

problem of deciding whether a given 3-CNF formula $\phi$ is satisfiable is one of the most known NP-complete problems. We can assume without loss of generality that each clause has literals that correspond to three distinct variables. Given the formula $\phi$, we construct in polynomial time a trapezoid graph $G_\phi$, such that $G_\phi$ is a triangle graph if and only if $\phi$ is satisfiable. Before constructing the whole trapezoid graph $G_\phi$, we construct first some smaller trapezoid graphs for each clause and each variable that appears in the given formula $\phi$.

## *6.1. The construction for each clause*

Consider a 3-CNF formula $\phi = \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k$ with $k$ clauses $\alpha_1, \alpha_2, \ldots, \alpha_k$ and $n$ boolean variables $x_1, x_2, \ldots, x_n$, such that $\alpha_i = (\ell_{i,1} \vee \ell_{i,2} \vee \ell_{i,3})$ for $i = 1, 2, \ldots, k$. For the literals $\ell_{i,1}, \ell_{i,2}, \ell_{i,3}$ of the clause $\alpha_i$, let $\ell_{i,1} \in \{x_{r_{i,1}}, \overline{x_{r_{i,1}}}\}$, $\ell_{i,2} \in \{x_{r_{i,2}}, \overline{x_{r_{i,2}}}\}$, and $\ell_{i,3} \in \{x_{r_{i,3}}, \overline{x_{r_{i,3}}}\}$, where $1 \le r_{i,1} < r_{i,2} < r_{i,3} \le n$. Let $L_1$ and $L_2$ be two parallel lines in the plane. To every clause $\alpha_i$, where $i = 1, 2, \ldots, k$, we associate the trapezoid representation $R_{\alpha_i}$ with 7 trapezoids that is illustrated in Fig. 7. Note that the trapezoid of the vertex $z_i$ in $R_{\alpha_i}$ is trivial, i.e. a line. In this construction, the trapezoids of the vertices $v_{i,1}$, $v_{i,2}$, and $v_{i,3}$ correspond to the literals $\ell_{i,1}$, $\ell_{i,2}$, and $\ell_{i,3}$, respectively. Furthermore, by the construction of $R_{\alpha_i}$, the left line of $v_{i,j}$ lies completely to the left of the left line of $v_{i,j+1}$ in $R_{\alpha_i}$ for $j \in \{1, 2\}$.

We prove now two basic properties of the construction $R_{\alpha_i}$ in Fig. 7 for the clause $\alpha_i$ that will be then used in the proof of correctness of our reduction.

**Lemma 7.** *Let $G_{\alpha_i}$ be the trapezoid graph induced by the trapezoid representation $R_{\alpha_i}$ of Fig. 7. Then, in* any *trapezoid representation $R$ of $G_{\alpha_i}$, such that $v_{i,1} \ll_R z_i$, one of $v_{i,1}, v_{i,2}, v_{i,3}$ is right-closed in $R$.*

**Proof.** Let $R$ be a trapezoid representation of $G_{\alpha_i}$, such that $v_{i,1} \ll_R z_i$. Note that $v_{i,2}, v_{i,3} \in N(v_{i,1}) \setminus N(z_i)$. Thus, since $v_{i,1} \ll_R z_i$ by assumption, it follows that also $v_{i,2} \ll_R z_i$ and $v_{i,3} \ll_R z_i$. Furthermore, note that $v'_{i,1} \in N(z_i) \setminus N(v_{i,1})$, $v'_{i,2} \in N(z_i) \setminus N(v_{i,2})$, and $v'_{i,3} \in N(z_i) \setminus N(v_{i,3})$. Therefore, since $v_{i,1} \ll_R z_i$, $v_{i,2} \ll_R z_i$, and $v_{i,3} \ll_R z_i$, it follows that $v_{i,1} \ll_R v'_{i,1}$, $v_{i,2} \ll_R v'_{i,2}$, and $v_{i,3} \ll_R v'_{i,3}$. Moreover, note that we can locally change appropriately in $R$ the right lines of $v_{i,1}, v_{i,2}, v_{i,3}$ and the left lines of $v'_{i,1}, v'_{i,2}, v'_{i,3}$, such that the relative position of the endpoints $R(v_{i,1}), R(v_{i,2}), R(v_{i,3})$ on the line $L_1$ is arbitrary. Therefore, we assume throughout the proof without loss of generality that $R(v_{i,1}) <_R R(v_{i,2}) <_R R(v_{i,3})$ (cf. Fig. 7).

We will now prove that $r(v_{i,3}) <_R r(v_{i,2}) <_R r(v_{i,1})$. Suppose otherwise that $r(v_{i,1}) <_R r(v_{i,2})$. Then, since $R(v_{i,1}) <_R R(v_{i,2})$ and $v_{i,2} \ll_R v'_{i,2}$ by the previous paragraph, it follows that also $v_{i,1} \ll_R v'_{i,2}$. This is a contradiction, since $v_{i,1}v'_{i,2} \in E(G_{\alpha_i})$ (cf. Fig. 7). Therefore $r(v_{i,2}) <_R r(v_{i,1})$. Now suppose that $r(v_{i,2}) <_R r(v_{i,3})$. Then, similarly, since $R(v_{i,2}) <_R R(v_{i,3})$ and $v_{i,3} \ll_R v'_{i,3}$ by the previous paragraph, it follows that also $v_{i,2} \ll_R v'_{i,3}$. This is again a contradiction, since $v_{i,2}v'_{i,3} \in E(G_{\alpha_i})$. Summarizing, $r(v_{i,3}) <_R r(v_{i,2}) <_R r(v_{i,1})$ (cf. Fig. 7).

Recall that $v_{i,1} \ll_R v'_{i,1}$. Therefore, since $r(v_{i,2}) <_R r(v_{i,1})$ by the previous paragraph, it follows that $r(v_{i,2}) <_R r(v_{i,1}) <_R r(v'_{i,1})$, i.e. $r(v_{i,2}) <_R l(v'_{i,1})$. Now, if $R(v_{i,2}) <_R L(v'_{i,1})$, then $v_{i,2} \ll_R v'_{i,1}$, which is a contradiction, since $v_{i,2}v'_{i,1} \in E(G_{\alpha_i})$ (cf. Fig. 7). Thus $L(v'_{i,1}) <_R R(v_{i,2})$. Summarizing, there exist two vertices $v_{i,1}, v'_{i,1} \in N(v_{i,2})$, such that $v_{i,1} \ll_R v'_{i,1}$, $L(v'_{i,1}) <_R R(v_{i,2})$, and $r(v_{i,2}) <_R l(v'_{i,1})$, and thus $v_{i,2}$ is upper-right-closed in $R$ by Definition 1. Therefore, since also $R(v_{i,2}) <_R R(v_{i,3})$ and $r(v_{i,3}) <_R r(v_{i,2})$, it follows that $L(v'_{i,1}) <_R R(v_{i,3})$ and $r(v_{i,3}) <_R l(v'_{i,1})$. Thus, since $v_{i,1}, v'_{i,1} \in N(v_{i,3})$, it follows by Definition 1 that also $v_{i,3}$ is upper-right-closed in $R$.

Similarly, since $v_{i,3} \ll_R v'_{i,3}$ and $R(v_{i,2}) <_R R(v_{i,3})$, it follows that $R(v_{i,2}) <_R R(v_{i,3}) <_R L(v'_{i,3})$, i.e. $R(v_{i,2}) <_R L(v'_{i,3})$. Now, if $r(v_{i,2}) <_R l(v'_{i,3})$, then $v_{i,2} \ll_R v'_{i,3}$, which is a contradiction, since $v_{i,2}v'_{i,3} \in E(G_{\alpha_i})$. Thus $l(v'_{i,3}) <_R r(v_{i,2})$. Summarizing, there exist two vertices $v_{i,3}, v'_{i,3} \in N(v_{i,2})$, such that $v_{i,3} \ll_R v'_{i,3}$, $l(v'_{i,3}) <_R r(v_{i,2})$, and $R(v_{i,2}) <_R L(v'_{i,3})$, and thus $v_{i,2}$ is lower-right-closed in $R$ by Definition 1. Therefore, since also $r(v_{i,2}) <_R r(v_{i,1})$ and $R(v_{i,1}) <_R R(v_{i,2})$, it follows that $l(v'_{i,3}) <_R r(v_{i,1})$ and $R(v_{i,1}) <_R L(v'_{i,3})$. Thus, since $v_{i,3}, v'_{i,3} \in N(v_{i,1})$, it follows by Definition 1 that also $v_{i,1}$ is lower-right-closed in $R$.

Summarizing, $v_{i,3}$ is upper-right-closed in $R$ and $v_{i,1}$ is lower-right-closed in $R$, while $v_{i,2}$ is both upper-right-closed and lower-right-closed in $R$, i.e. $v_{i,2}$ is right-closed in $R$ by Definition 2. This completes the proof of the lemma. $\square$

**Corollary 1.** *Consider the trapezoid representation $R_{\alpha_i}$ of Fig. 7. For every $p \in \{1, 2, 3\}$, we can locally change appropriately in $R_{\alpha_i}$ the right lines of $v_{i,1}, v_{i,2}, v_{i,3}$ and the left lines of $v'_{i,1}, v'_{i,2}, v'_{i,3}$, such that $v_{i,p}$ is right-closed and $v_{i,p'}$ is right-open, for every $p' \in \{1, 2, 3\} \setminus \{p\}$.*

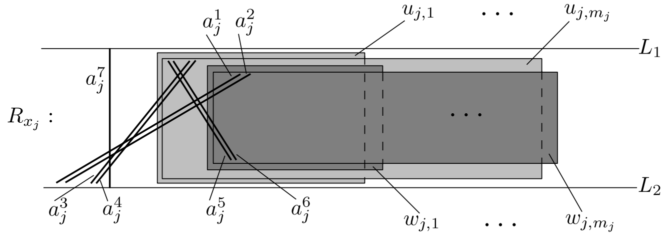


**Fig. 8.** The construction $R_{x_j}$ that is associated to the variable $x_j$ of the formula $\phi$, where $j = 1, 2, \ldots, n$.

**Proof.** Note that in the representation $R_{\alpha_i}$ of Fig. 7, the relative position of the endpoints $R(v_{i,1}), R(v_{i,2}), R(v_{i,3})$ on the line $L_1$ is $R(v_{i,1}) <_{R_{\alpha_i}} R(v_{i,2}) <_{R_{\alpha_i}} R(v_{i,3})$. Then, it follows by the proof of Lemma 7 that $v_{i,2}$ is right-closed in $R_{\alpha_i}$. Moreover, it is straightforward to see that the other two trapezoids $v_{i,1}$ and $v_{i,3}$ are right-open in $R_{\alpha_i}$ (in particular, $v_{i,1}$ is upper-right-open in $R_{\alpha_i}$ and $v_{i,3}$ is lower-right-open in $R_{\alpha_i}$).

Furthermore, recall by the proof of Lemma 7 that we can locally change appropriately in $R_{\alpha_i}$ the right lines of $v_{i,1}, v_{i,2}, v_{i,3}$ and the left lines of $v'_{i,1}, v'_{i,2}, v'_{i,3}$, such that the relative position of the endpoints $R(v_{i,1}), R(v_{i,2}), R(v_{i,3})$ on the line $L_1$ is arbitrary. For an arbitrary $p \in \{1, 2, 3\}$, consider now the trapezoid representation $R$ that is obtained by changing locally these lines in $R_{\alpha_i}$, such that the endpoint $R(v_{i,p})$ lies in the middle of $R(v_{i,1}), R(v_{i,2}), R(v_{i,3})$ on $L_1$. Then, similarly to the above, $v_{i,p}$ is right-closed in this representation $R$, while the other two trapezoids $v_{i,p'}$ are right-open in $R$, for every $p' \in \{1, 2, 3\} \setminus \{p\}$. This completes the proof of the corollary. □

### 6.2. The construction for each variable

Let $x_j$ be a variable of the formula $\phi$, where $1 \le j \le n$. Let $x_j$ appear in $\phi$ (either as $x_j$ or negated as $\overline{x_j}$) in the $m_j$ clauses $\alpha_{i_{j,1}}, \alpha_{i_{j,2}}, \ldots, \alpha_{i_{j,m_j}}$, where $1 \le i_{j,1} < i_{j,2} < \cdots < i_{j,m_j} \le k$. Then, we associate to the variable $x_j$ the trapezoid representation $R_{x_j}$ with $2m_j + 7$ trapezoids that is illustrated in Fig. 8. In this construction, the trapezoids of the vertices $u_{j,t}$ and $w_{j,t}$, where $1 \le t \le m_j$, correspond to the appearance of the variable $x_j$ (either as $x_j$ or negated as $\overline{x_j}$) in the clause $\alpha_{i_{j,t}}$ in $\phi$. Note that the trapezoids of the vertices $a_j^1, a_j^2, \ldots, a_j^7$ are trivial, i.e. lines. By the construction of $R_{x_j}$, the right line of $u_{j,t}$ lies completely to the left of the right line of $w_{j,t}$ for all values of $j = 1, 2, \ldots, n$ and $t = 1, 2, \ldots, m_j$. Furthermore, the right line of each of $\{u_{j,t}, w_{j,t}\}$ lies completely to the left of the right line of each of $\{u_{j,t'}, w_{j,t'}\}$ in $R_{x_j}$, whenever $t < t'$.

### 6.3. The construction the trapezoid graph $G_\phi$

We construct now a trapezoid representation $R_\phi$ of the whole trapezoid graph $G_\phi$, by composing the constructions $R_{\alpha_i}$ and $R_{x_j}$ presented in Sections 6.1 and 6.2, as follows. First, we place in $R_\phi$ the $k$ trapezoid representations $R_{\alpha_i}$, where $i = 1, 2, \ldots, k$, between the lines $L_1$ and $L_2$ such that, whenever $i < i'$, every trapezoid of $R_{\alpha_i}$ lies completely to the left of every trapezoid of $R_{\alpha_{i'}}$. Then, we place in $R_\phi$ the $n$ trapezoid representations $R_{x_j}$, where $j = 1, 2, \ldots, n$, between the lines $L_1$ and $L_2$ such that, whenever $j < j'$, the lines of $a_j^1, a_j^2, \ldots, a_j^7$ and the left lines of all $u_{j,t}, w_{j,t}$, lie completely to the left of the lines of $a_{j'}^1, a_{j'}^2, \ldots, a_{j'}^7$ and the left lines of all $u_{j',t'}, w_{j',t'}$. Moreover, for every $j, j' = 1, 2, \ldots, n$, the lines of $a_j^1, a_j^2, \ldots, a_j^7$ and the left lines of all $u_{j,t}, w_{j,t}$, lie in $R_\phi$ completely to the left of the right lines of all $u_{j',t'}, w_{j',t'}$. Thus, note in particular that every $u_{j,t}$ intersects every other $u_{j',t'}$ and every $w_{j',t'}$ in $R_\phi$.

Let $j \in \{1, 2, \ldots, n\}$ and $t \in \{1, 2, \ldots, m_j\}$. Recall that, by the construction of $R_{x_j}$ in Section 6.2, the pair of trapezoids $\{u_{j,t}, w_{j,t}\}$ corresponds to the appearance of the variable $x_j$ in a clause $\alpha_i$ of $\phi$, where $i = i_{j,t} \in \{1, 2, \ldots, k\}$. That is, either $\ell_{i,p} = x_j$ or $\ell_{i,p} = \overline{x_j}$ for some $p \in \{1, 2, 3\}$, where $\alpha_i = (\ell_{i,1} \vee \ell_{i,2} \vee \ell_{i,3})$. Then, we place in $R_\phi$ the right lines of the trapezoids $u_{j,t}$ and $w_{j,t}$ directly before the left line of $v_{i,p}$ (i.e. no line of any other trapezoid intersects with or lies between the right lines of $u_{j,t}$ and $w_{j,t}$ and the left line of $v_{i,p}$).

In order to finalize the construction of $R_\phi$, we distinguish now the two cases regarding the literal $\ell_{i,p}$ of the clause $\alpha_i$, in which the variable $x_j$ appears. If $\ell_{i,p} = x_j$, then we add to $R_\phi$ six trivial trapezoids (i.e. lines) $\{b_{j,t}^1, b_{j,t}^2, \ldots, b_{j,t}^6\}$, as it is shown in Fig. 9(b). On the other hand, if $\ell_{i,p} = \overline{x_j}$, then we add to $R_\phi$ eight trivial trapezoids (i.e. lines) $\{b_{j,t}^1, b_{j,t}^2, \ldots, b_{j,t}^8\}$, as it is shown in Fig. 9(b). In particular, we place these six (resp. eight) new lines in $R_\phi$ such that they intersect only the right lines of $u_{j,t}$ and $w_{j,t}$ and the left line of $v_{i,p}$ in $R_\phi$. Note that the trapezoid graphs induced by the representations in Fig. 9(a) and (b) are isomorphic to the graphs investigated in Lemmas 5 and 6, respectively. This completes the construction of the trapezoid representation $R_\phi$, while $G_\phi$ is the trapezoid graph induced by $R_\phi$.

It is now easy to verify that, by the construction of $R_\phi$, all the trapezoids $u_{j,t}$ are upper-left-closed and right-closed in $R_\phi$, while all the trapezoids $w_{j,t}$ are lower-right-closed and left-closed in $R_\phi$. Furthermore, all the trapezoids $u_{j,t}$ are lower-left-open in $R_\phi$ and all the trapezoids $w_{j,t}$ are upper-right-open in $R_\phi$. Consider now a trapezoid $v_{i,p}$ in $R_\phi$. If $v_{i,p}$ corresponds to a positive literal $\ell_{i,p} = x_j$ (for some variable $x_j$), then $v_{i,p}$ is upper-left-closed and lower-left-open in $R_\phi$ (cf. Fig. 9(a)). On the other hand, if $v_{i,p}$ corresponds to a negative literal $\ell_{i,p} = \overline{x_j}$, then $v_{i,p}$ is left-closed in $R_\phi$ (cf. Fig. 9(b)).

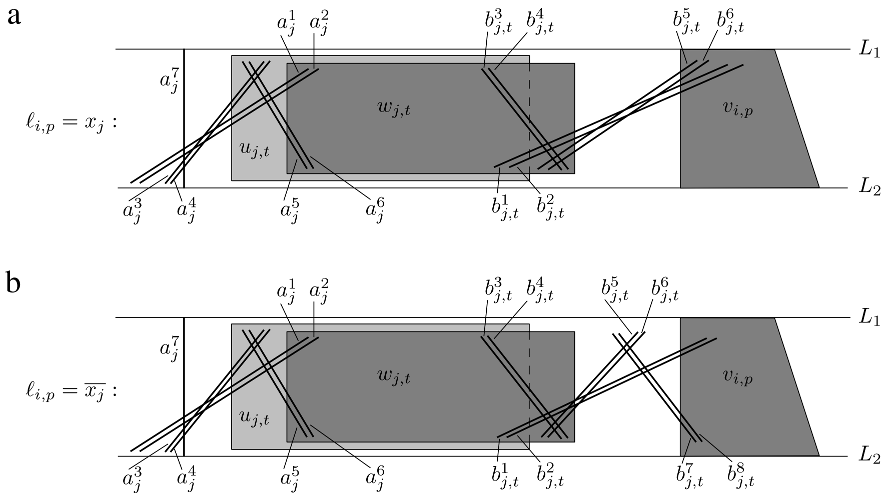


**Fig. 9.** The composition of the trapezoids of $R_{x_j}$ with the trapezoid $v_{i,p}$ of $R_{\alpha_i}$, in the cases where (a) $\ell_{i,p} = x_j$ and (b) $\ell_{i,p} = \overline{x_j}$.

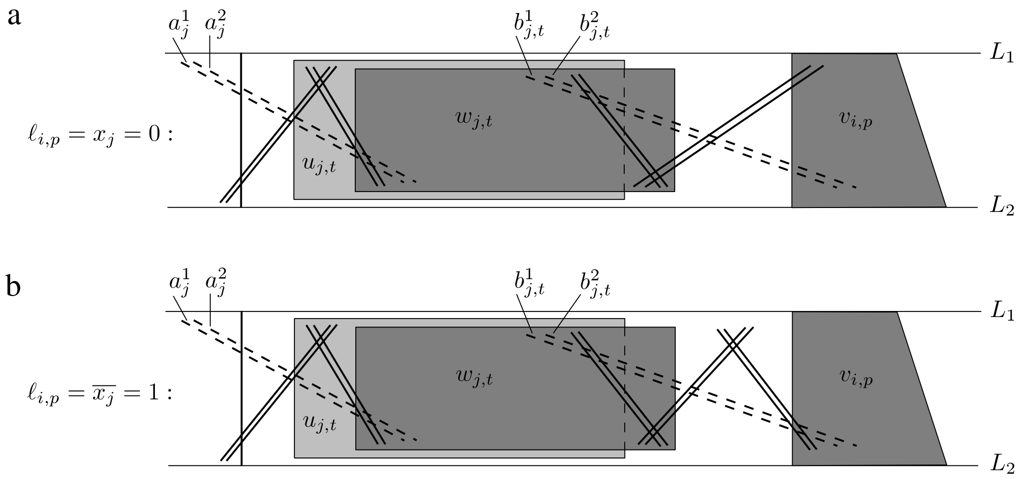


**Fig. 10.** The case where $x_j = 0$ in the truth assignment $\tau$: the horizontal axis flipping operations of the lines $\{a_j^2, a_j^2\}$ and $\{b_{j,t}^2, b_{j,t}^2\}$, $t = 1, 2, \ldots, m_j$, where (a) $\ell_{i,p} = x_j = 0$, (b) $\ell_{i,p} = \overline{x_j} = 1$.

In order to prove the correctness of our reduction (cf. Theorem 3), we prove separately the necessary and sufficient conditions in the next two lemmas.

**Lemma 8.** *If the formula $\phi$ is satisfiable, then $G_\phi$ is a triangle graph.*

**Proof.** Suppose that $\phi$ has a satisfying truth assignment $\tau$. Starting from $R_\phi$, we will construct a standard trapezoid representation $R_0$ of $G_\phi$. This will then imply that $G_\phi$ is a triangle graph by Theorem 1.

First consider an index $j$ that corresponds to a variable $x_j = 0$ in the truth assignment $\tau$. Furthermore consider an imaginary line $L_3$ that is parallel to $L_1$ and $L_2$ and has the same distance from both $L_1$ and $L_2$. We replace in $R_\phi$ the lines $\{a_j^1, a_j^2\}$ (resp. the lines $\{b_{j,t}^1, b_{j,t}^2\}$ for every index $t = 1, 2, \ldots, m_j$) by their mirror image along $L_3$, such that, in the resulting representation, these flipped lines intersect with the same trapezoids as the lines $\{a_j^1, a_j^2\}$ (resp. the lines $\{b_{j,t}^1, b_{j,t}^2\}$) intersect in $R_\phi$. In the case where the corresponding literal $\ell_{i,p}$ equals a variable $x_j$, i.e. if $\ell_{i,p} = x_j = 0$, these flipping operations are illustrated in Fig. 10(a). Otherwise, in the case where the corresponding literal $\ell_{i,p}$ equals a negated variable $\overline{x_j}$, i.e. if $\ell_{i,p} = \overline{x_j} = 1$, these flipping operations are illustrated in Fig. 10(b). For better visibility, the flipped lines are drawn dashed in Fig. 10. For every other index $j$ that corresponds to a variable $x_j = 1$ in $\tau$, we leave the lines $\{a_j^1, a_j^2\}$, as well as all the lines $\{b_{j,t}^1, b_{j,t}^2\}$, at the same position in $R_0$ as in $R_\phi$.

Note that, after performing these flipping operations, for every index $j$ that corresponds to a variable $x_j = 0$ in $\tau$, all the trapezoids $u_{j,t}$ are left-closed and right-open, while all the trapezoids $w_{j,t}$ are left-open and right-closed. On the contrary, for every index $j$ that corresponds to a variable $x_j = 1$ in $\tau$, all the trapezoids $u_{j,t}$ are left-open and right-closed, while all the trapezoids $w_{j,t}$ are left-closed and right-open (cf. Fig. 9). That is, after performing the above flipping operations, for every variable $x_j$ of $\phi$, all the trapezoids $u_{j,t}$ and $w_{j,t}$ are open in the resulting trapezoid representation. Moreover, for all indices $i, p$, the trapezoid $v_{i,p}$ is left-open in the resulting trapezoid representation if and only if the literal $\ell_{i,p}$ is satisfied in $\tau$, i.e. if and only if $\ell_{i,p} = 1$ in $\tau$ (cf. Figs. 9 and 10).

Let us now complete the construction of $R_0$ from $R_\phi$. Consider first a clause $\alpha_i$ of $\phi$, where $i = 1, 2, \ldots, k$. Then, $\alpha_i$ has at least one satisfied literal $\ell_{i,p} = 1$ in the truth assignment $\tau$, where $p \in \{1, 2, 3\}$, since $\tau$ is assumed to be a satisfying assignment of $\phi$. Therefore, after performing the above flipping operations, there exists by the previous paragraph at least one trapezoid $v_{i,p}$, where $p \in \{1, 2, 3\}$, which is left-open in the resulting trapezoid representation. Recall now by Corollary 1 that we can locally change in $R_{\alpha_i}$ the right lines of $v_{i,1}, v_{i,2}, v_{i,3}$ and the left lines of $v'_{i,1}, v'_{i,2}, v'_{i,3}$, such that $v_{i,p}$ is right-closed and $v_{i,p'}$ is right-open, for every $p' \in \{1, 2, 3\} \setminus \{p\}$. Therefore, after changing appropriately these lines of the trapezoids, $v_{i,p}$ is right-closed and left-open in the resulting representation, while $v_{i,p'}$ is right-open, for every $p' \in \{1, 2, 3\} \setminus \{p\}$. That is, all $v_{i,1}, v_{i,2}, v_{i,3}$ are open in the resulting trapezoid representation.

Denote by $R_0$ the trapezoid representation that is obtained if we perform all the above local changes. Note that all trapezoids $u_{j,t}$, $w_{j,t}$, and $v_{i,p}$ are open in $R_0$ for all pairs of indices $j, t$ and $i, p$. Furthermore, it is easy to see that also the trapezoids $v'_{i,p}$ are right-open in $R_0$ (in particular, they are also right-open in the initial trapezoid representation $R_{\alpha_i}$, cf. Fig. 7). All the remaining trapezoids of $R_0$ are trivial, i.e. lines, and thus also trivial triangles. Therefore, $R_0$ is a standard trapezoid representation of $G_\phi$ by Definition 3, and thus $G_\phi$ is a triangle graph by Theorem 1. This completes the proof of the lemma. □

**Lemma 9.** *If $G_\phi$ is a triangle graph, then the formula $\phi$ is satisfiable.*

**Proof.** Suppose that $G_\phi$ is a triangle graph and let $R$ be a triangle representation of $G_\phi$. We construct a truth assignment $\tau$ of the variables $x_1, x_2, \ldots, x_n$ that satisfies the formula $\phi$, as follows. For any $j = 1, 2, \ldots, n$ such that $a_j^7 \ll_R u_{j,1}$, we define $x_j = 1$ if and only if $u_{j,1}$ is left-open in $R$. Similarly, for any $j = 1, 2, \ldots, n$ such that $u_{j,1} \ll_R a_j^7$, we define $x_j = 1$ if and only if $u_{j,1}$ is right-open in $R$. We will prove that the truth assignment $\tau$ satisfies $\phi$.

Let $i \in \{1, 2, \ldots, k\}$. By possibly performing a vertical axis flipping of $R$, we may assume without loss of generality that $v_{j,1} \ll_R z_i$. Therefore, since $v_{i,2}, v_{i,3} \in N(v_{i,1}) \setminus N(z_i)$, it follows that also $v_{i,2} \ll_R z_i$ and $v_{i,3} \ll_R z_i$. Consider now the subgraph $H_0$ of $G_\phi$ induced by the vertices $\{v_{i,1}, v_{i,2}, v_{i,3}, v'_{i,1}, v'_{i,2}, v'_{i,3}, z_i\}$. Note that $H_0$ is isomorphic to the graph induced by the trapezoid representation $R_{\alpha_i}$ of Fig. 7. Then, Lemma 7 implies that one of $v_{i,1}, v_{i,2}, v_{i,3}$ is right-closed in the restriction $R[H_0]$ of $R$. Let in the following $v_{i,p}$ be right-closed in $R[H_0]$, and thus also right-closed in $R$, for some $p \in \{1, 2, 3\}$. Thus $L(v'_{i,p'}) <_R R(v_{i,p}) <_R L(v'_{i,p''})$ and $l(v'_{i,p''}) <_R r(v_{i,p}) <_R l(v'_{i,p'})$, for appropriate values of the indices $p', p'' \in \{1, 2, 3\} \setminus \{p\}$, cf. Fig. 7.

Furthermore, let $x_j$ be the variable that appears in the literal $\ell_{i,p}$ of the clause $\alpha_i$, i.e. either $\ell_{i,p} = x_j$ or $\ell_{i,p} = \overline{x_j}$. Moreover, let the trapezoids $\{u_{j,t}, w_{j,t}\}$ correspond to the appearance of the variable $x_j$ in $\ell_{i,p}$, for some index $t \in \{1, 2, \ldots, m_j\}$. Note that, since the trapezoids of vertices $u_{j,t}$ and $v_{i,p}$ do not intersect in $R$, it follows that either $u_{j,t} \ll_R v_{i,p}$ or $v_{i,p} \ll_R u_{j,t}$. We will prove that $u_{j,t} \ll_R v_{i,p}$. Suppose otherwise that $v_{i,p} \ll_R u_{j,t}$. Then, since $u_{j,t} \notin N(z_i)$, it follows that either $v_{i,p} \ll_R z_i \ll_R u_{j,t}$ or $v_{i,p} \ll_R u_{j,t} \ll_R z_i$. Let first $v_{i,p} \ll_R z_i \ll_R u_{j,t}$. Then, since $b_{j,t}^1 \in N(v_{i,p})$ and $b_{j,t}^1 \in N(u_{j,t})$ by the construction of $G_\phi$ (cf. Fig. 9(a) and (b)), it follows that the line of $b_{j,t}^1$ intersects the line of $z_i$ in $R$. This is a contradiction, since $b_{j,t}^1 \notin N(z_i)$ by the construction of $G_\phi$. Let now $v_{i,p} \ll_R u_{j,t} \ll_R z_i$ and let $q \in \{1, 2, 3\} \setminus \{p\}$. Then, since $v'_{i,q} \in N(v_{i,p})$ and $v'_{i,q} \in N(z_i)$ by the construction of $G_\phi$ (cf. Fig. 7), it follows that the trapezoid of $v'_{i,q}$ intersects the trapezoid of $u_{j,t}$ in $R$. This is again a contradiction, since $v'_{i,q} \notin N(u_{j,t})$ by the construction of $G_\phi$. Therefore $u_{j,t} \ll_R v_{i,p}$.

Now, we will now prove that $a_j^7 \ll_R u_{j,t}$. Suppose otherwise that $u_{j,t} \ll_R a_j^7$. Then, since $a_j^7 \notin N(v_{i,p})$, it follows that either $u_{j,t} \ll_R v_{i,p} \ll_R a_j^7$ or $u_{j,t} \ll_R a_j^7 \ll_R v_{i,p}$. Let first $u_{j,t} \ll_R v_{i,p} \ll_R a_j^7$. Then, since $a_j^1 \in N(u_{j,t})$ and $a_j^1 \in N(a_j^7)$ by the construction of $G_\phi$ (cf. Fig. 9(a) and (b)), it follows that the line of $a_j^1$ intersects the trapezoid of $v_{i,p}$ in $R$. This is a contradiction, since $a_j^1 \notin N(v_{i,p})$ by the construction of $G_\phi$. Let now $u_{j,t} \ll_R a_j^7 \ll_R v_{i,p}$. Then, since $b_{j,t}^1 \in N(u_{j,t})$ and $b_{j,t}^1 \in N(v_{i,p})$ by the construction of $G_\phi$ (cf. Fig. 7), it follows that the line of $b_{j,t}^1$ intersects the line of $a_j^7$ in $R$. This is again a contradiction, since $b_{j,t}^1 \notin N(a_j^7)$ by the construction of $G_\phi$. Therefore $a_j^7 \ll_R u_{j,t}$.

That is, $a_j^7 \ll_R u_{j,t} \ll_R v_{i,p} \ll_R z_i$. Note here by the construction of $G_\phi$, that the existence of the lines $\{a_j^1, a_j^2, \ldots, a_j^7\}$ guarantees that $u_{j,1}$ is left-open in $R$ if and only if $u_{j,t}$ is left-open in $R$, for any $t = 2, \ldots, m_j$ (cf. Fig. 8). Therefore, due to the truth assignment $\tau$ of the variables $x_1, x_2, \ldots, x_n$ that we defined above, it follows that $x_j = 1$ if and only if $u_{j,t}$ is left-open in $R$, for any $t = 1, 2, \ldots, m_j$. We distinguish in the following the two cases regarding the literal $\ell_{i,p}$ of the clause $\alpha_i$.

Let first $\ell_{i,p} = x_j$. Consider the subgraph $H_1$ of $G_\phi$ induced by the vertices $\{u_{j,t}, w_{j,t}, v_{i,p}\} \cup \{a_j^1, \ldots, a_j^7\} \cup \{b_{j,t}^1, b_{j,t}^2, \ldots, b_{j,t}^6\}$. Note that $H_1$ is isomorphic to the graph induced by the trapezoid representation of Fig. 6(a). Furthermore, consider the subgraph $H_2$ of $G_\phi$ induced by the vertices $V(H_1) \cup \{v'_{i,p'}, v'_{i,p''}\}$, where $\{p', p''\} = \{1, 2, 3\} \setminus \{p\}$. Note now by the construction of $G_\phi$ that the existence of the lines $\{a_j^1, a_j^2, \ldots, a_j^7\}$ guarantees that the trapezoid $u_{j,t}$ is left-open in $R$ if and only if $u_{j,t}$ is left-open in the restriction $R[H_1]$ of $R$.

Moreover, the connected graph $H_2$ satisfies the conditions of Theorem 2. Indeed, $v_{i,p}$ is a cut vertex of $H_2$ and $H_2 - \{v_{i,p}\}$ has the two connected components $H_1$ and $G_\phi[v'_{i,p'}, v'_{i,p''}]$. Therefore, since $R[H_2]$ is a triangle representation, Theorem 2 implies that $v_{i,p}$ is open in $R[H_2]$. Recall now that $L(v'_{i,p'}) <_R R(v_{i,p}) <_R L(v'_{i,p''})$ and $l(v'_{i,p''}) <_R r(v_{i,p}) <_R l(v'_{i,p'})$, for appropriate values of the indices $p', p'' \in \{1, 2, 3\} \setminus \{p\}$. Therefore, since $b_{j,t}^1 \ll_R v'_{i,p'}$ and $b_{j,t}^1 \ll_R v'_{i,p''}$, it follows by Definitions 1 and 2 that $v_{i,p}$ is right-closed in $R[H_2]$. Thus, since $v_{i,p}$ is open in $R[H_2]$, it follows that $v_{i,p}$ is left-open in $R[H_2]$. Therefore, $v_{i,p}$ is also left-open in $R[H_1]$, since $H_1$ is an induced subgraph of $H_2$. Now, Lemma 5 implies that $u_{j,t}$ is left-open in $R[H_1]$, since $a_j^7 \ll_R u_{j,t}$ and $v_{i,p}$ is left-open in $R[H_1]$. Therefore $u_{j,t}$ is also left-open in $R$, and thus it follows by the definition of the truth assignment $\tau$ that $\ell_{i,p} = x_j = 1$.

Let now $\ell_{i,p} = \overline{x_j}$. Consider the subgraph $H_3$ of $G_\phi$ induced by the vertices $\{u_{j,t}, w_{j,t}, v_{i,p}\} \cup \{a_j^1, \ldots, a_j^7\} \cup \{b_{j,t}^1, b_{j,t}^2, \ldots, b_{j,t}^8\}$. Note that $H_3$ is isomorphic to the graph induced by the trapezoid representation of Fig. 6(b). Furthermore, consider the subgraph $H_4$ of $G_\phi$ induced by the vertices $V(H_3) \cup \{v'_{i,p'}, v'_{i,p''}\}$, where $\{p', p''\} = \{1, 2, 3\} \setminus \{p\}$.

Moreover, the connected graph $H_4$ satisfies the conditions of Theorem 2. Indeed, $v_{i,p}$ is a cut vertex of $H_4$ and $H_4 - \{v_{i,p}\}$ has the two connected components $H_3$ and $G_\phi[v'_{i,p'}, v'_{i,p''}]$. Therefore, since $R[H_4]$ is a triangle representation, Theorem 2 implies that $v_{i,p}$ is open in $R[H_4]$. Recall that $L(v'_{i,p'}) <_R R(v_{i,p}) <_R L(v'_{i,p''})$ and $l(v'_{i,p''}) <_R r(v_{i,p}) <_R l(v'_{i,p'})$, for appropriate values of the indices $p'$ and $p''$, where $\{p', p''\} = \{1, 2, 3\} \setminus \{p\}$. Therefore, since $b_{j,t}^1 \ll_R v'_{i,p'}$ and $b_{j,t}^1 \ll_R v'_{i,p''}$, it follows by Definitions 1 and 2 that $v_{i,p}$ is right-closed in $R[H_4]$. Thus, since $v_{i,p}$ is open in $R[H_4]$, it follows that $v_{i,p}$ is left-open in $R[H_4]$. Therefore, $v_{i,p}$ is also left-open in $R[H_3]$, since $H_3$ is an induced subgraph of $H_4$. Now, Lemma 6 implies that $u_{j,t}$ is left-closed in $R[H_3]$, since $a_j^7 \ll_R u_{j,t}$ and $v_{i,p}$ is left-open in $R[H_3]$. Therefore $u_{j,t}$ is also left-closed in $R$, and thus, it follows by the definition of the truth assignment $\tau$ that $x_j = 0$, i.e. $\ell_{i,p} = \overline{x_j} = 1$.

Summarizing, for an arbitrary index $i \in \{1, 2, \ldots, k\}$, we proved that there exists an index $p \in \{1, 2, 3\}$, such that the literal $\ell_{i,p}$ is satisfied by the truth assignment $\tau$, i.e. $\ell_{i,p} = 1$. Therefore, every clause $\alpha_i$, where $i \in \{1, 2, \ldots, k\}$, is satisfied by $\tau$, and thus the whole formula $\phi$ is satisfied by $\tau$. This completes the proof of the lemma. □

The next theorem follows now directly by Lemmas 8 and 9.

**Theorem 3.** *The formula $\phi$ is satisfiable if and only if $G_\phi$ is a triangle graph.*

Therefore, since 3SAT is NP-complete, Theorem 3 implies that the recognition of triangle graphs is NP-hard. Moreover, since the recognition of triangle graphs lies in NP by Observation 1, and since $G_\phi$ is a trapezoid graph, we can summarize our main result in the next theorem.

**Theorem 4.** *Given a graph G, it is NP-complete to decide whether G is a triangle graph. The problem remains NP-complete even if the given graph G is known to be a trapezoid graph.*

## 7. Concluding remarks

In this article we proved that the triangle graph (known also as *PI** graph) recognition problem is NP-complete, by providing a reduction from the 3SAT problem, thus answering a longstanding open question. Our reduction implies that this problem remains NP-complete even in the case where the input graph is a trapezoid graph. The recognition of *simple-triangle* graphs [3], as well as the recognition of the related classes of *unit* and *proper tolerance* graphs [11,1] (these are subclasses of bounded tolerance, i.e. parallelogram, graphs [1]), *proper bitolerance* graphs [11,2] (they coincide with *unit bitolerance* graphs [2]), and *multitolerance* graphs [19] (they naturally generalize trapezoid graphs [23,19]) remain interesting open problems for further research.

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