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The Friendship Problem on Graphs*

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In this paper we provide a purely combinatorial proof of the Friendship Theorem, which has been first proven by P. Erdös et al. by using also algebraic methods. Moreover, we generalize this theorem in a natural way, assuming that every pair of nodes occupies $\ell \geq 2$ common neighbors. We prove that every graph, which satisfies this generalized ℓ*-friendship condition*, is a regular graph.

Keywords: Friendship theorem, friendship graph, windmill graph, Kotzig's conjecture.

1 INTRODUCTION

A graph is called a *friendship graph* if every pair of its nodes has exactly one common neighbor. This condition is called the *friendship condition*. Furthermore, a graph is called a *windmill graph*, if it consists of $k \geq 1$ triangles, which have a unique common node, known as the "politician". Clearly, any windmill graph is a friendship graph. Erdös *et al.* [1] were the first who proved the Friendship Theorem on graphs:

Theorem 1 (Friendship Theorem). *Every friendship graph is a windmill graph.*

The proof of Erdös *et al.* used both combinatorial and algebraic methods [1]. Due to the importance of this theorem in various disciplines and applications except graph theory, such as in the field of block designs and

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coding theory [2], as well as in the set theory [3], several different approaches have been used to provide a simpler proof.

In 1971, Wilf provided a geometric proof of the Friendship Theorem by using projective planes [4], while in 1972, Longyear and Parsons gave a proof by counting neighbors, walks and cycles in regular graphs [3]. Both Longyear *et al.* and Wilf refer to an unpublished proof of G. Higman in lecture form at a conference on combinatorics in 1969; however, to the best of our knowledge, no known printed article of this proof exists. Hammersley avoided the use of eigenvalues and provided in 1983 a proof using numerical techniques [5]. He extended the Friendship Theorem to the so called "love problem", where self loops are allowed. In 2001, Aigner and Ziegler mentioned the Friendship Theorem in [6] as one of the greatest theorems of Erdös of all time. In the same year, West gave a proof similar to that in [3], counting common neighbors and cycles [7]. Finally, Huneke gave in 2002 two proofs, one being more combinatorial and one that combines combinatorics and linear algebra [8].

The friendship condition can be rewritten as follows: "For every pair of nodes, there is exactly one path of length two between them". In this direction, the friendship problem can be generalized as follows: *Find all graphs, in which every pair of nodes is connected with exactly ℓ paths of length k*. Such graphs are called *ℓ-regularly k-path connected* graphs, or simply $P_\ell(k)$-graphs [9]. The Friendship Theorem implies that the $P_1(2)$-graphs are exactly the windmill graphs. For the case of $P_1(k)$-graphs, where $k > 2$, Kotzig conjectured in 1974 that there exists no such graph (*Kotzig's conjecture*) [10] and he proved this conjecture for $3 \leq k \leq 8$ [11]. Kostochka proved in 1988 that the conjecture is true for $k \leq 20$ [12]. Furthermore, Xing and Hu proved the Kotzig's conjecture in 1994 for $k \geq 12$ [13] and Yang *et al.* in 2000 for the cases $k = 9, 10$ and 11 [14]. Thus, the Kotzig's conjecture is valid now as a theorem.

In Section 2 of this paper we propose a simple purely combinatorial proof of the Friendship Theorem. At first step, we prove that any graph G satisfying the friendship condition is a windmill graph, under the assumption that G has at least one node of degree at most two. At second step, we prove that G is a regular graph in the case that all its nodes have degree greater than two. Finally, we prove by contradiction that G has always a node of degree two, following a counting argument similar to [3].

In Section 3, we generalize the friendship condition in a natural way to the *ℓ-friendship condition*: "Every pair of nodes has exactly $\ell \geq 2$ common neighbors". The graphs that satisfy the ℓ-friendship condition are exactly the $P_\ell(2)$-graphs and they are called *ℓ-friendship graphs*. We prove that every ℓ-friendship graph is a regular graph, for every $\ell \geq 2$. This result implies that the ℓ-friendship graphs coincide with the class of *strongly regular graphs*

$srg(n, k, \lambda, \mu)$ with $\lambda = \mu = \ell$, which correspond to symmetric balanced incomplete block designs [7]. This class of graphs has been extensively studied and several non-trivial examples of them are known in the literature [15, 16]. Finally, in Section 4 we summarize the results obtained in this paper.

2 A COMBINATORIAL PROOF OF THE FRIENDSHIP THEOREM

In this section we propose a purely combinatorial proof of the Friendship Theorem, i.e. that every friendship graph is a windmill graph. In the following, denote by C_4 a node-simple cycle on 4 nodes, by $N(v)$ the set of neighbors of v in G and $N[v] = N(v) \cup \{v\}$.

Lemma 1. *Let G be a friendship graph. Then G is connected and it contains no C_4 as a subgraph. Furthermore* $\deg(v) \geq 2$ *for every node v of G, and the distance between any two nodes in G is at most two.*

Proof. The proof is done by contradiction. If G is not connected, then there are at least two nodes of G with no common neighbor, which is in contradiction to the friendship condition. If G includes C_4 as a subgraph (not necessary induced), there are two nodes v and u with at least two common neighbors, as it is illustrated in Figure 1(a). This is a contradiction to the friendship condition. Assume that $\deg(v) = 1$ for a node v of G, and let u be the unique neighbor of v. Then, v has no common neighbor with u, which is again a contradiction. Finally, if a pair (v, u) of G has distance at least three, then v and u have no common neighbor in G, which is also a contradiction.

Since $\deg(v) \geq 2$ for every node v of a friendship graph G by Lemma 1, we may distinguish the nodes of a friendship graph by their degree, as Definition 1 states.

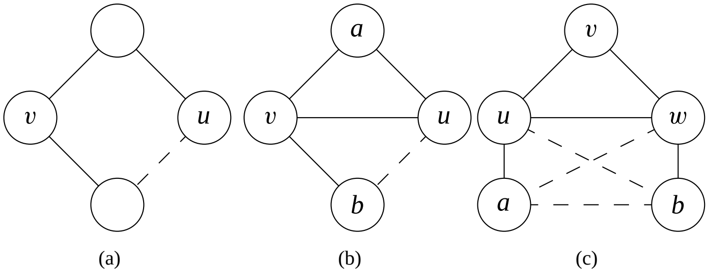

FIGURE 1
Three forbidden cases.

Definition 1. *In a friendship graph G, every node v with* $\deg(v) = 2$ *is called a* simple node, *otherwise it is called a* complex node.

Lemma 2. *For every node v of a friendship graph G, $N[v]$ induces a windmill graph.*

Proof. Consider two nodes v and $u \in N(v)$. Due to the assumption, they have a unique common neighbor a, as it is illustrated in Figure 1(b). Consider now another node $b \in N(v) \setminus \{u, a\}$. If $b \in N(u)$, then G includes a C_4 as a subgraph, which is a contradiction due to Lemma 1. Thus, $b \notin N(u)$. Since this holds for every node $b \in N(v) \setminus \{u, a\}$, it follows that every node $u \in N(v)$ produces with v exactly one triangle. Therefore, for every node v of G, $N[v]$ induces a windmill graph.

Lemma 3. *If a friendship graph G has at least one simple node, then G is a windmill graph.*

Proof. Consider a simple node v of G with $N(v) = \{u, w\}$, as it is illustrated in Figure 1(c). Due to Lemma 2, u and w are also neighbors. At first, since u and w have a unique common neighbor, all their neighbors are distinct, except v. In the case where G is constituted of only these three nodes, G is obviously a windmill graph. Otherwise, every node of $V \setminus \{v, u, w\}$ is either a neighbor of u or of w, since in the opposite case it would have no common neighbor with v, which is a contradiction. Finally, consider two nodes $a \in N(u) \setminus \{v, w\}$ and $b \in N(w) \setminus \{v, u\}$. Then, a and b are not neighbors, since otherwise u, w, b and a would induce a C_4, which is in contradiction to Lemma 1. It follows that the distance between a and b is three, which is also a contradiction. Thus, at least one node of $\{u, w\}$ is simple and the other one is neighbored to all other nodes in G. It follows that G is a windmill graph, due to Lemma 2.

Lemma 4. *If a friendship graph G has no simple node, then G is a $2k$-regular graph with $2k(2k-1)+1$ nodes, for some $k \geq 2$.*

Proof. Suppose that all nodes of G are complex nodes, i.e. their degree is greater than two. Let v be such a node of G. Then, all the remaining nodes in $V \setminus \{v\}$ are partitioned into the sets $L = N(v)$ and $L' = V \setminus N[v]$.

Due to Lemma 2 and the assumption, $N[v]$ induces a non-trivial windmill graph, as it is illustrated in Figure 2. Suppose now that the windmill graph $N[v]$ has $k \geq 2$ triangles. Thus the graph induced by $N(v)$ is a perfect matching of size k with edges: $\{v_1^0, v_1^1\}, \{v_2^0, v_2^1\}, \ldots, \{v_k^0, v_k^1\}$. Now consider a node v_i^x of L , for some $i \in \{1, 2, \ldots, k\}$ and $x \in \{0, 1\}$. Denote

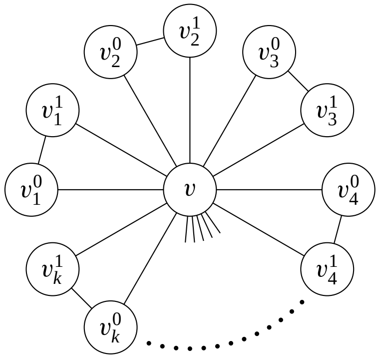

FIGURE 2
A non-trivial windmill graph.

by $N'(v_i^x) = N(v_i^x) \cap L'$ the set of nodes of the windmill graph $N[v_i^x]$ that belong to L', as it is illustrated in Figure 3. Due to the assumption it follows that $N'(v_i^x) \neq \emptyset$.

Due to the windmill structure of $N[v_i^x]$, $N'(v_i^x)$ constitutes a perfect matching of $k_i^x \geq 1$ pairs of nodes in L', denoted by $P_\ell(v_i^x)$, $\ell = 1, 2, \ldots, k_i^x$. Clearly, there is no edge connecting two nodes from two different pairs $P_a(v_i^x)$ and $P_b(v_i^x)$, since otherwise there exists a C_4, which is a contradiction due to Lemma 1. Similarly, an arbitrary node in $N'(v_i^x)$ does not have any other neighbor in L except v_i^x, since otherwise there exists again a C_4. Define now the i^{th} *block* $B_i := N'(v_i^0) \cup N'(v_i^1)$, as it is illustrated in Figure 3.

Since $k \geq 2$, there are at least two different blocks B_i and B_j in G. Consider now a node $q \in N'(v_j^0)$, as it is illustrated in Figure 4. Since the nodes q and v_i^0 have exactly one common neighbor, q has exactly one neighbor p in $N'(v_i^0)$. On the other hand, the only neighbor of p in $N'(v_j^0)$ is q, since otherwise p would have more than one common neighbor with v_j^0, which is a contradiction. Thus, the edges between $N'(v_i^0)$ and $N'(v_j^0)$ constitute a perfect matching. This holds similarly for the edges between $N'(v_i^x)$ and $N'(v_j^y)$ as well, where $x, y \in \{0, 1\}$ and hence, it holds $k_i^0 = k_i^1 =: k'$ for every $i \in \{1, 2, \ldots, k\}$.

Now, an arbitrary node $p \in N'(v_i^0)$ is a neighbor to *exactly* two nodes q and s of any of the $k - 1$ blocks B_j, $j \neq i$, one in $N'(v_j^0)$ and one in $N'(v_j^1)$, as it is illustrated in Figure 4. Similarly, q and s are neighbors to exactly two nodes q' and s' of $N'(v_i^1)$, respectively. Therefore, since p has a common neighbor with every node of $N'(v_i^1)$, it follows that $2(k-1) \geq |N'(v_i^1)| = 2k'$. If $2(k-1) > 2k'$, then there exist two neighbors q, s of p in $\bigcup_{j \neq i} B_j$, such that both q and s have the same neighbor $z \in N'(v_i^1)$. Thus G contains a

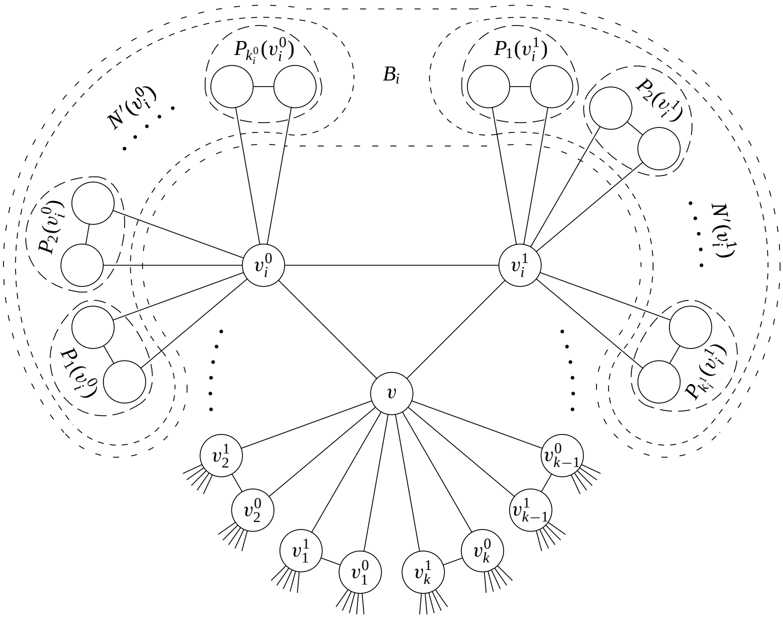

FIGURE 3
The i^{th} block B_i.

C_4 on the vertices p, q, s, z, which is a contradiction by Lemma 1. Therefore $2(k-1) = 2k'$, i.e. $k' = k-1$. Thus, taking into account the two neighbors r and u_i^0 of p, it has exactly $2(k-1)+2 = 2k$ neighbors in G. Furthermore, any node v_i^x has $2k'+2 = 2k$ neighbors in G as well. Thus, since $\deg(v) = 2k$, it follows that G is a $2k$-regular graph. Finally, since the blocks B_i, $i \in \{1, 2, \ldots, k\}$ have $2k \cdot 2(k-1)$ nodes in total and since v has $2k$ neighbors, it follows that G has $n = 2k(2k-1)+1$ nodes.

Lemma 5. *There is at least one simple node in any friendship graph G.*

Proof. The proof will be done by contradiction. Suppose that all nodes of G are complex, i.e. their degree is greater than two. Then, by Lemma 4, G is a $2k$-regular graph with $n = 2k(2k-1)+1$ nodes, for some $k \geq 2$. For an arbitrary natural number $\ell \geq 2$, let $T(\ell)$ be the set of all ordered ℓ-tuples $\langle v_1, v_2, \ldots, v_\ell \rangle$ of (not necessary distinct) nodes of G, such that v_i is neighbored with v_{i+1} for every $i \in \{1, 2, \ldots, \ell-1\}$. Since $n = 2k(2k-1)+1$, it holds that

$$|T(\ell)| = n \cdot (2k)^{\ell-1} \equiv 1 \bmod (2k-1) \qquad (1)$$

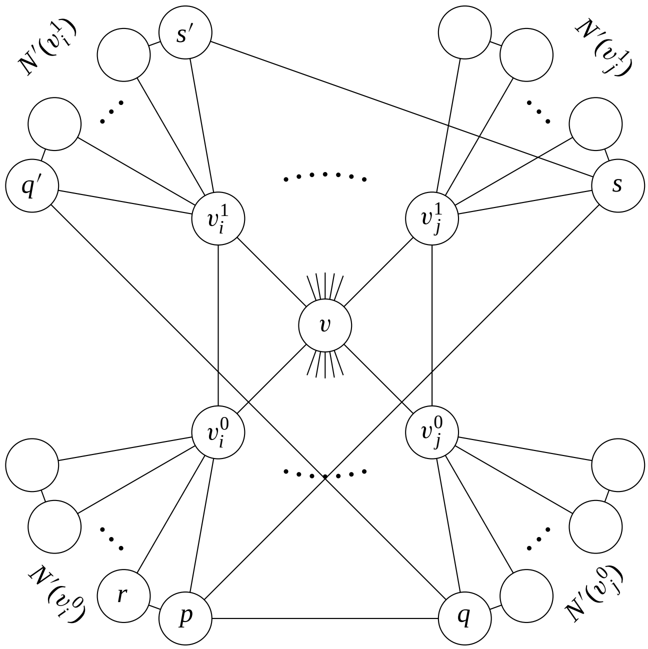

FIGURE 4
The regularity of the friendship graph G.

for every $\ell \geq 2$. If the nodes v_ℓ and v_1 are neighbored, then the tuple $\langle v_1, v_2, \ldots, v_\ell \rangle$ constitutes a *closed ℓ-walk* in G. Let $C(\ell) \subseteq T(\ell)$ be the set of all closed ℓ-walks. Let furthermore $C^*(\ell) = \{\langle v_1, v_2, \ldots, v_{\ell-1}, v_\ell \rangle \in T(\ell) : v_\ell = v_1\}$ be the set of all closed $(\ell - 1)$-walks in G.

Consider now the surjective mapping $f : C(\ell) \to T(\ell - 1)$, such that $f(\langle v_1, v_2, \ldots, v_{\ell-1}, v_\ell \rangle) = \langle v_1, v_2, \ldots, v_{\ell-1} \rangle$. For every tuple $\langle v_1, v_2, \ldots, v_{\ell-1} \rangle$ of $T(\ell - 1) \setminus C^*(\ell - 1)$, i.e. with $v_{\ell-1} \neq v_1$, it holds that $\langle v_1, v_2, \ldots, v_{\ell-1} \rangle = f(\langle v_1, v_2, \ldots, v_{\ell-1}, y \rangle)$, where y is the unique common neighbor of $v_{\ell-1}$ and v_1 in G. On the other hand, for every tuple $\langle v_1, v_2, \ldots, v_{\ell-1} = v_1 \rangle$ of $C^*(\ell - 1)$ it holds that $\langle v_1, v_2, \ldots, v_{\ell-1} = v_1 \rangle = f(\langle v_1, v_2, \ldots, v_{\ell-1} = v_1, z \rangle)$, where z is any of the $2k$ neighbors of v_1 in G. Since f is surjective and due to (1), it follows that

$$\begin{aligned} |C(\ell)| &= 2k \cdot |C^*(\ell - 1)| + |T(\ell - 1) \setminus C^*(\ell - 1)| \\ &\equiv |T(\ell - 1)| \bmod (2k - 1) \qquad (2) \\ &\equiv 1 \bmod (2k - 1) \end{aligned}$$

for every $\ell \geq 2$.

Now, for an arbitrary prime divisor p of $2k-1$, consider the bijective mapping (cyclic permutation) $\pi : C(p) \to C(p)$, with $\pi(\langle v_1, v_2, \ldots, v_p\rangle) = \langle v_2, \ldots, v_p, v_1\rangle$. Since p is a prime number, all tuples $\pi^i(\langle v_1, v_2, \ldots, v_p\rangle)$, where $i \in \{1, 2, \ldots, p\}$ are distinct. The mapping π defines in a trivial way an equivalence relation: the tuples $\langle v_1, v_2, \ldots, v_p\rangle$ and $\langle w_1, w_2, \ldots, w_p\rangle$ are equivalent if there is a number $t \in \{1, 2, \ldots, p\}$, such that $\pi^t(\langle v_1, v_2, \ldots, v_p\rangle) = \langle w_1, w_2, \ldots, w_p\rangle$. This equivalence relation partitions $C(p)$ into equivalence classes of p elements each and thus, it holds that

$$|C(p)| \equiv 0 \bmod (p) \tag{3}$$

Since p is a prime divisor of $2k-1$, (3) is in contradiction to (2) for $\ell = p$.

The Friendship Theorem follows now directly from to Lemmas 2, 3, 4 and 5.

3 THE GENERALIZED FRIENDSHIP PROBLEM

In this section we generalize the friendship condition, assuming that each pair of nodes occupies exactly $\ell \geq 2$ common neighbors. We prove that these graphs are d-regular, with $d \geq \ell + 1$.

Definition 2. *The condition: "Every pair of nodes has exactly ℓ common neighbors" is called the* ℓ-friendship condition. *The graphs that satisfy the ℓ-friendship condition are exactly the $P_\ell(2)$-graphs and they are called ℓ-*friendship graphs.

Proposition 1. *Every ℓ-friendship graph G is a regular graph, for $\ell \geq 2$.*

Proof. Consider a node $v \in V$ with $d = \deg(v)$. Similarly to Section 2, denote $L = N(v)$ and $L' = V \setminus N[v]$. Obviously, every node of the set L' has distance 2 from v. Consider now a node $a \in L$. It follows that a has exactly ℓ neighbors in L, since the pair $\{v, a\}$ has exactly ℓ common neighbors in G.

Suppose at first that $L' = \emptyset$. Let $L \cap N(a) = \{a_1, a_2, \ldots, a_\ell\}$. For every $i \in \{1, 2, \ldots, \ell\}$, the pair $\{a, a_i\}$ has v as a common neighbor and $\ell - 1$ more common neighbors in L. It follows that $a_i \in N(a_j)$ for every $i \neq j \in \{1, 2, \ldots, \ell\}$, i.e. the tuple $\{v, a, a_1, \ldots, a_\ell\}$ constitutes an $(\ell+2)$-clique, as it is illustrated in Figure 5. Now, suppose that $L \setminus \{a, a_1, a_2, \ldots, a_\ell\} \neq \emptyset$ and consider a node $b \in L \setminus \{a, a_1, a_2, \ldots, a_\ell\}$. This node has no neighbor in the

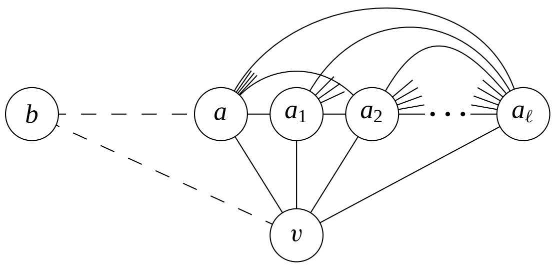

FIGURE 5
The case $L' = \emptyset$.

set $\{a, a_1, a_2, \ldots, a_\ell\}$, since otherwise at least one node of this set would have more than ℓ neighbors in L, which is a contradiction. Thus, the pair $\{a, b\}$ has v as the only common neighbor, which is also a contradiction, since $\ell \geq 2$. Therefore, if $L' = \emptyset$, then G is isomorphic to the complete graph $K_{\ell+1}$ and therefore G is an $(\ell + 1)$-regular graph.

Suppose now that $L' \neq \emptyset$. As it is illustrated in Figure 6, every node $x \in L'$ has exactly ℓ neighbors in L, since otherwise the pair $\{v, x\}$ would not have exactly ℓ common neighbors in G. If we fix the node $a \in L$, then there exist in G exactly $(d - 1)\,\ell$ paths of length two with extreme nodes a and b, where $b \in L$, since there are $d - 1$ nodes $b \in L \setminus \{a\}$ and every such pair $\{a, b\}$ has exactly ℓ common neighbors in G. Among them, exactly $d - 1$ ones have v as the intermediate node. Furthermore, exactly $\ell\,(\ell - 1)$ ones have their intermediate node in L, since a has exactly ℓ neighbors in L and each of them has $\ell - 1$ other neighbors in L except a. Thus, each of the remaining

$$(d - 1)\,\ell - (d - 1) - \ell\,(\ell - 1) = (d - \ell - 1)\,(\ell - 1)$$

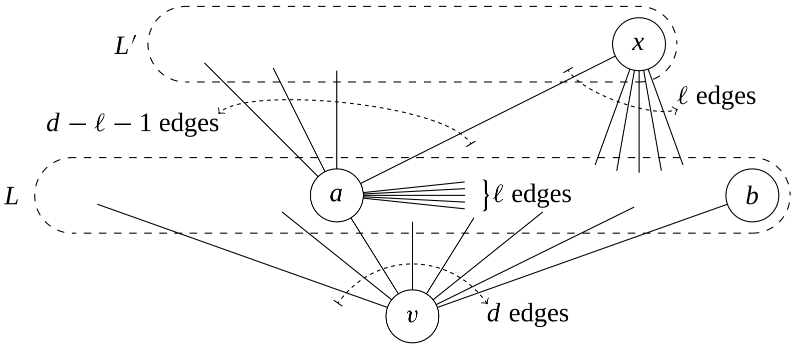

FIGURE 6
The case $L' \neq \emptyset$.

paths has a node in L' as their intermediate node. Consider now a node $x \in L' \cap N(a)$. The edge between a and x is included in exactly $\ell - 1$ paths of length two with extreme nodes a and b, where $b \in L$, since x has exactly $\ell - 1$ other neighbors in L except a. Thus, every $a \in L$ is neighbored to exactly

$$\frac{(d-\ell-1)(\ell-1)}{(\ell-1)} = (d-\ell-1) \tag{4}$$

nodes in L'. It follows that

$$\left|L'\right| = \frac{d(d-\ell-1)}{\ell} \tag{5}$$

since L includes d nodes, each one of them has $d-\ell-1$ neighbors in L' and each node of L' is neighbored to ℓ nodes of L. Finally, since $|V| = |L| + \left|L'\right| + 1$ and $|L| = d$, it follows from (5) that

$$|V| = \frac{d(d-1)}{\ell} + 1 \tag{6}$$

Since (6) holds for the degree d of an arbitrary node $v \in V$, it results that every node v has equal degree d in G and therefore G is a d-regular graph.

A graph G with n nodes is called a *strongly regular* graph if there exist parameters k, λ, μ such that G is k-regular, every pair of adjacent nodes have exactly λ common neighbors, and every pair of non-adjacent nodes has exactly μ common neighbors [7]. The class of strongly regular graphs with n nodes and parameters k, λ, μ is denoted by $srg(n, k, \lambda, \mu)$. Due to Proposition 1, the *ℓ-friendship graphs* coincide with the strongly regular graphs $srg(n, k, \lambda, \mu)$ with $\lambda = \mu = \ell$. Several non-trivial examples of $srg(n, k, \ell, \ell)$ are known in the literature, e.g. the line graph of K_6 with $n = 15, k = 8, \ell = 4$ [16], the cartesian product $K_4 \times K_4$ (or Shrikhande graph) with $n = 16, k = 6, \ell = 2$ and the halved 5-cube graph with $n = 16, k = 10, \ell = 6$, which is referred to as Clebsch graph in [15].

4 CONCLUSION

In this paper we propose a purely combinatorial proof of the Friendship Theorem, originally proved by Erdös *et al.* Furthermore, we generalize the simple friendship condition in a natural way to the *ℓ-friendship condition*: "Every pair of nodes has exactly $\ell \geq 2$ common neighbors" and we prove that every graph which satisfies this condition is a regular graph. It remains open to characterize fully this class of graphs, which together with the recent proof of the Kotzig's conjecture, will complete the characterization of the graphs $P_\ell(2)$ and $P_1(k)$ that are the direct generalizations of the class $P_1(2)$ of the friendship graphs.

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