An Intersection Model for Multitolerance Graphs: Efficient Algorithms and Hierarchy

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Abstract

Tolerance graphs model interval relations in such a way that intervals can tolerate a certain degree of overlap without being in conflict. This class of graphs has attracted many research efforts, mainly due to its interesting structure and its numerous applications, especially in DNA sequence analysis and resource allocation, among others. In one of the most natural generalizations of tolerance graphs, namely *multitolerance* graphs, two tolerances are allowed for each interval – one from the left and one from the right side of the interval. Then, in its interior part, every interval tolerates the intersection with others by an amount that is a convex combination of its two border-tolerances. In the comparison of DNA sequences between different organisms, the natural interpretation of this model lies on the fact that, in some applications, we may want to treat several parts of the genomic sequences differently. That is, we may want to be more tolerant at some parts of the sequences than at others. These two tolerances for every interval – together with their convex hull – define an infinite number of the so called *tolerance-intervals*, which make the multitolerance model inconvenient to cope with. In this article we introduce the first non-trivial intersection model for multitolerance graphs, given by objects in the 3-dimensional space called *trapezoepipeds*. Apart from being important on its own, this new intersection model proves to be a powerful tool for designing efficient algorithms. Given a multitolerance graph with n vertices and m edges, we present algorithms that compute a minimum coloring and a maximum clique in optimal $O(n \log n)$ time, and a maximum weight independent set in $O(m + n \log n)$ time. Moreover, our results imply an optimal $O(n \log n)$ time algorithm for the maximum weight independent set problem on tolerance graphs, thus closing the complexity gap for this problem. Additionally, by exploiting more the new 3D-intersection model, we completely classify multitolerance graphs in the hierarchy of perfect graphs.

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1 Introduction.

A graph $G = (V, E)$ on n vertices is a *tolerance* graph if there exists a collection $I = \{I_v \mid v \in V\}$ of closed intervals on the real line and a set $t = \{t_v \mid v \in V\}$ of positive numbers, such that for any two vertices $u, v \in V$, $uv \in E$ if and only if $|I_u \cap I_v| \geq \min\{t_u, t_v\}$, where $|I|$ denotes the length of the interval I. The pair $\langle I, t\rangle$ is called a *tolerance representation* of G. If G has a tolerance representation $\langle I, t\rangle$, such that $t_v \leq |I_v|$ for every $v \in V$, then G is called a *bounded tolerance* graph and $\langle I, t\rangle$ a *bounded tolerance representation* of G.

Tolerance graphs have been introduced in [8], in order to generalize some of the well known applications of interval graphs. If in the definition of tolerance graphs we replace the operation "min" between tolerances by "max", we obtain the class of *max-tolerance* graphs. Both tolerance and max-tolerance graphs have attracted many research efforts [2, 4, 5, 9–11, 15, 16, 20, 21] as they find numerous applications, especially in bioinformatics, constrained-based temporal reasoning, and resource allocation problems, among others [10,11,15,16]. In particular, one of their applications is in the comparison of DNA sequences from different organisms or individuals by making use of a software tool like BLAST [1].

BLAST takes a special genomic sequence Q as a parameter and returns all sequences from its comprehensive database that share a strong similarity with at least some part of the input query sequence Q. The returned sequences from BLAST, together with the corresponding parts of Q with which they share a strong similarity, can be viewed as a tolerance or a max-tolerance graph (depending on the interpretation of "strong similarity"). Moreover, a subset of the returned sequences which share with each other a certain part of Q, are said to build a *cluster*. Such maximal clusters are exactly the maximal cliques of the corresponding tolerance (or max-tolerance) graph; it turns out that these clusters can be interpreted as functional domains carrying biologically meaningful information. There exist efficient algorithms that output all (at most $O(n^3)$) possible maximal cliques of a max-tolerance graph [15, 16], while the number of maximal cliques in a tolerance graph may be exponential [10].

In some circumstances, we may want to treat different parts of the above genomic sequences in BLAST non-uniformly, since for instance some of them may be biologically less significant or we have less confidence in the exact sequence due to sequencing errors in more error prone genomic regions. That is, we may want to be more tolerant at some parts of the sequences than at others. This concept leads naturally to the notion of *multitolerance* (known also as *bitolerance*) graphs [11, 22]. The main idea is to allow two different tolerances l_t and r_t to each interval, one to the left and one to the right side, respectively. Then, every interval tolerates in its interior part the intersection with other intervals by an amount that is a convex combination of l_t and r_t.

Formally, let $I = [l, r]$ be a closed interval on the real line and $l_t, r_t \in I$ be two numbers between l and r, called *tolerant points*. For every $\lambda \in [0, 1]$, we define the interval $I_{l_t,r_t}(\lambda) = [l + (r_t - l)\lambda,\ l_t + (r - l_t)\lambda]$, which is the convex combination of $[l, l_t]$ and $[r_t, r]$. Furthermore, we define the set $\mathcal{I}(I, l_t, r_t) = \{I_{l_t,r_t}(\lambda) \mid \lambda \in [0, 1]\}$ of intervals. That is, $\mathcal{I}(I, l_t, r_t)$ is the set of all intervals that we obtain when we linearly transform $[l, l_t]$ into $[r_t, r]$. For an interval I, the *set of tolerance-intervals* τ of I is defined either as $\tau = \mathcal{I}(I, l_t, r_t)$ for some values $l_t, r_t \in I$ of tolerant points, or as $\tau = \{\mathbb{R}\}$. A graph $G = (V, E)$ is a *multitolerance* graph if there exists a collection $I = \{I_v \mid v \in V\}$ of closed intervals and a family $t = \{\tau_v \mid v \in V\}$ of sets of tolerance-intervals, such that: for any two vertices $u, v \in V$, $uv \in E$ if and only if there exists an element $Q_u \in \tau_u$ with $Q_u \subseteq I_v$, or there exists an element $Q_v \in \tau_v$ with $Q_v \subseteq I_u$. Then, the pair $\langle I, t\rangle$ is called a *multitolerance representation* of G. Note that, in general, the adjacency of two vertices u and v in a multitolerance graph G depend on both sets of tolerance-intervals τ_u and τ_v. However, since the real line $\mathbb{R}$ is not included in any finite interval, if $\tau_u = \{\mathbb{R}\}$ for some vertex u of G, then the adjacency of u with another vertex v of G depends *only* on the set of tolerance-intervals τ_v of v. If G has a multitolerance representation $\langle I, t\rangle$, in which $\tau_v \neq \{\mathbb{R}\}$ for every $v \in V$, then G is called a *bounded multitolerance* graph and $\langle I, t\rangle$ a *bounded multitolerance representation* of G.

A graph $G = (V, E)$ with n vertices is the *intersection graph* of a family $F = \{S_1, \ldots, S_n\}$ of subsets of a set S if there exists a bijection $\mu : V \to F$ such that for any two distinct vertices $u, v \in V$, $uv \in E$ if and only if $\mu(u) \cap \mu(v) \neq \emptyset$. Then, F is called an *intersection model* of G. Note that every graph has a trivial intersection model based on adjacency relations [18]. Note also that a multitolerance representation is not an intersection model, since two intervals may intersect without the corresponding vertices being necessarily adjacent. Some intersection models provide a natural and intuitive understanding of the structure of a class of graphs, and turn out to be very helpful in the design of efficient algorithms that solve optimization problems [18]. Therefore, it is of great importance to establish non-trivial intersection models for families of graphs. In particular, many important graph classes can be described as intersection graphs of set families that are derived from some kind of geometric configuration.

For instance, a *permutation* (resp. *parallelogram* and *trapezoid*) graph is the intersection graph of line segments (resp. parallelograms and trapezoids) between two parallel lines L_1 and L_2 [7]. Such a representation with line segments (resp. parallelograms and trapezoids) is called a *permutation* (resp. *parallelogram* and *trapezoid*) *representation* of this graph. Recently, two natural intersection models for max-tolerance graphs [15] and for tolerance graphs [20] have been presented, given by semi-squares on the plane [15] and by parallelepipeds in the 3-dimensional space [20], respectively. These two representations have been used to design efficient algorithms for several generally NP-hard optimization problems on tolerance and max-tolerance graphs, see [15, 16, 20].

Bounded multitolerance graphs (also known as *bounded bitolerance* graphs [3, 11, 14]) coincide with trapezoid graphs [11, 22], which have received considerable attention in the literature, see [11]. However, the intersection model of trapezoids between two parallel lines can not cope with general multitolerance graphs, in which the set τ_v of tolerance-intervals for a vertex v can be $\tau_v = \{\mathbb{R}\}$. Therefore, the only way until now to deal with general multitolerance graphs was to use the inconvenient multitolerance representation, which uses an infinite number of tolerance-intervals. This is the main reason why, despite their apparent practical interpretation, only little is known about multitolerance graphs, e.g. that the minimum fill-in problem can be solved efficiently and that the difference between the pathwidth and the treewidth is at most one [22].

Our contribution In this article we introduce the first non-trivial intersection model for general multitolerance graphs, given by objects in the 3-dimensional space, called *trapezoepipeds*. This *trapezoepiped representation* unifies in a simple and intuitive way the widely known trapezoid representation for bounded multitolerance graphs and the parallelepiped representation for tolerance graphs [20]. The main idea is to exploit the third dimension to capture the information of the vertices with $\tau_v = \{\mathbb{R}\}$ as the set of tolerance-intervals. This intersection model can be constructed in linear time given a multitolerance representation.

Apart of being important on its own, the trapezoepiped representation can be also used to design efficient algorithms. Given a multitolerance graph with n vertices and m edges, we present algorithms that compute a minimum coloring and a maximum clique in $O(n \log n)$ time (which turns out to be *optimal*), and a maximum weight independent set in $O(m + n \log n)$ time (where $\Omega(n \log n)$ is a lower bound for the complexity of this problem [6]). Moreover, we present a variation of the latter algorithm that computes a maximum weight independent set in optimal $O(n \log n)$ time, when the input is a tolerance graph, thus closing the complexity gap of [20]. Note here that, although the parallelepiped representation of tolerance graphs is similar to the trapezoepiped representation of multitolerance graphs, the coloring and clique algorithms presented in [20] do not extend to the case of multitolerance graphs, and thus the here presented algorithms are new. On the contrary, the algorithm presented in [20] for the maximum weight independent set with complexity $O(n^2)$ on tolerance graphs can be extended with the same time complexity to the case of multitolerance graphs; nevertheless we present here new algorithms for this problem that achieve better running times $O(m + n \log n)$ for multitolerance graphs and optimal $O(n \log n)$ for tolerance graphs.

Moreover, we prove several structural results on the class of multitolerance graphs, using our new intersection model and some known results from the hierarchy of perfect graphs given in [11]. In particular, we prove that multitolerance graphs *strictly* include tolerance and trapezoid graphs, as well as that they are *strictly* included in weakly chordal and in co-perfectly orderable graphs. Furthermore, we prove that multitolerance graphs are incomparable with alternately orientable and cocomparability graphs, i.e. none of these classes includes the other one. These results complement the hierarchy of perfect graphs given in [11]. The resulting hierarchy of classes of perfect graphs is *complete*, i.e. all inclusions are strict.

Notation In this article we follow standard notation and terminology, see for instance [11]. We consider finite, simple, and undirected graphs. Given a graph $G = (V, E)$, we denote by n the cardinality of V. An edge between vertices u and v is denoted by uv, and in this case vertices u and v are said to be *adjacent*. $\overline{G}$ denotes the *complement* of G, i.e. $\overline{G} = (V, \overline{E})$, where $uv \in \overline{E}$ if and only if $uv \notin E$. Given a subset of vertices $S \subseteq V$, the graph $G[S]$ denotes the graph *induced* by the vertices in S, i.e. $G[S] = (S, F)$, where for any two vertices $u, v \in S$, $uv \in F$ if and only if $uv \in E$. A subset $S \subseteq V$ is an *independent set* in G if the graph $G[S]$ has no edges. For a subset $K \subseteq V$, the induced subgraph $G[K]$ is a *complete subgraph* of G, or a *clique*, if each two of its vertices are adjacent. The maximum cardinality of a clique in G is denoted by $\omega(G)$ and is termed the *clique number* of G. A *proper coloring* of G is an assignment of different colors to adjacent vertices, which results in a partition of V into independent sets. The minimum number of colors for which there exists a proper coloring is denoted by $\chi(G)$ and is termed the *chromatic number* of G. A proper coloring of G with $\chi(G)$ colors, i.e. a partition of V into $\chi(G)$ independent sets, is a *minimum coloring* of G.

Organization of the paper We present the new intersection model for multitolerance graphs in Section 2. In Section 3 we present a canonical representation of multitolerance graphs and an algorithm that computes it in $O(n \log n)$ time. Then, using this algorithm, we present in Section 4 optimal $O(n \log n)$ time coloring and clique algorithms for multitolerance graphs. In Section 5 we present algorithms that compute a maximum weight independent set in $O(m + n \log n)$ time on a multitolerance graph, and in optimal $O(n \log n)$ time on a tolerance graph. In Section 6 we classify multitolerance graphs in the hierarchy of perfect graphs of [11]. Finally, we discuss the presented results and further research in Section 7.

2 An intersection model for multitolerance graphs.

In this section we present a 3D intersection model for general multitolerance graphs, which unifies the intersection model of trapezoids in the plane for bounded multitolerance graphs [11] and that of parallelepipeds in the 3-dimensional space for tolerance graphs [20]. Given a multitolerance graph $G = (V, E)$ along with a multitolerance representation $\langle I, t \rangle$ of G, recall that vertex $v \in V$ corresponds to an interval $I_v = [l_v, r_v]$ on the real line and a set τ_v of tolerance-intervals, where either $\tau_v = \mathcal{I}(I_v, l_{t_v}, r_{t_v})$ for some values $l_{t_v}, r_{t_v} \in I_v$ of tolerant points, or $\tau_v = \{\mathbb{R}\}$.

Definition 2.1. *Given a multitolerance representation of a multitolerance graph $G = (V, E)$, vertex $v \in V$ is* bounded *if $\tau_v = \mathcal{I}(I_v, l_{t_v}, r_{t_v})$ for some values $l_{t_v}, r_{t_v} \in I_v$. Otherwise, v is* unbounded. *V_B and V_U are the sets of bounded and unbounded vertices in V, respectively. Clearly $V = V_B \cup V_U$.*

Definition 2.2. *For a vertex $v \in V_B$ (resp. $v \in V_U$) in a multitolerance representation of G, the values $t_{v,1} = l_{t_v} - l_v$ and $t_{v,2} = r_v - r_{t_v}$ (resp. $t_{v,1} = t_{v,2} = \infty$) are the* left tolerance *and the* right tolerance *of v, respectively. Moreover, if $v \in V_U$, then $t_v = \infty$ is the* tolerance *of v.*

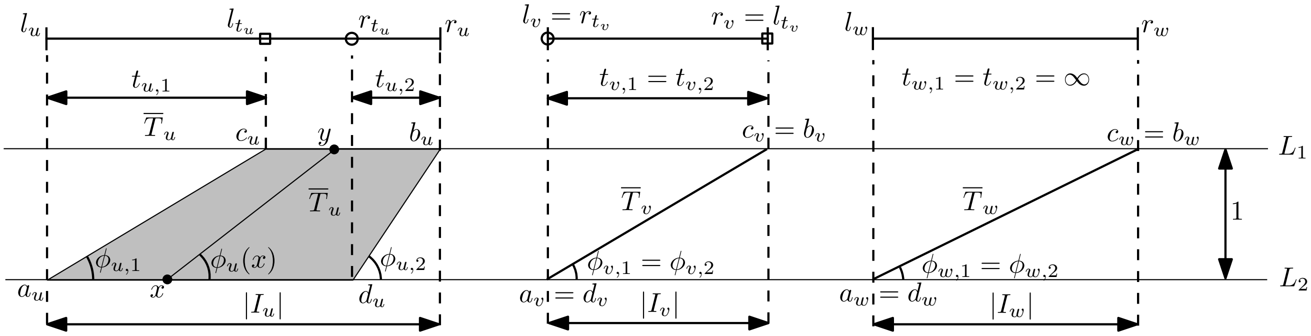

Figure 1: Trapezoids $\overline{T}_u$ and $\overline{T}_v$ correspond to bounded vertices u and v, respectively, while $\overline{T}_w$ corresponds to an unbounded vertex w.

It can be now easily seen by Definition 2.2 that if we set $t_{v,1} = t_{v,2}$ for every vertex $v \in V$, then we obtain a tolerance representation, in which $t_{v,1} = t_{v,2}$ is the (unique) tolerance of v. We may assume w.l.o.g. that no two bounded vertices share an endpoint or tolerant point, i.e. $\{l_u, r_u, l_{t_u}, r_{t_u}\} \cap \{l_v, r_v, l_{t_v}, r_{t_v}\} = \emptyset$ for all $u, v \in V_B$ with $u \neq v$ [22]. Furthermore, by possibly performing a small shift of the endpoints and the tolerant points, we may assume w.l.o.g. that $t_{v,1}, t_{v,2} > 0$ for every $v \in V$ and that the left and right tolerances for every bounded vertex are distinct, i.e. $\{t_{u,1}, t_{u,2}\} \cap \{t_{v,1}, t_{v,2}\} = \emptyset$ for all $u, v \in V_B$ with $u \neq v$. Similarly, if $t_{v,1} \neq |I_v|$ (resp. $t_{v,2} \neq |I_v|$) for a bounded vertex $v \in V_B$, we may assume w.l.o.g. that also $t_{v,2} \neq |I_v|$ (resp. $t_{v,1} \neq |I_v|$). That is, for every $v \in V_B$, either $t_{v,1} = t_{v,2} = |I_v|$, or $t_{v,1} < |I_v|$ and $t_{v,2} < |I_v|$. For more details in the cases of tolerance and bounded multitolerance graphs we refer to [11].

Let now L_1 and L_2 be two parallel lines at unit distance in the Euclidean plane. In the following we define for every vertex $v \in V$ a trapezoid $\overline{T}_v$ in the plane between the lines L_1 and L_2. The values $\tan \phi$ and $\cot \phi = \frac{1}{\tan \phi}$ denote the tangent and the cotangent of a slope ϕ, respectively. Furthermore, $\phi = arc\cot x$ is the slope ϕ, for which $\cot \phi = x$.

Definition 2.3. *Given an interval $I_v = [l_v, r_v]$ and tolerances $t_{v,1}, t_{v,2}$, $\overline{T}_v$ is the trapezoid in $\mathbb{R}^2$ defined by the points c_v, b_v on L_1 and a_v, d_v on L_2, where $a_v = l_v$, $b_v = r_v$, $c_v = \min\{r_v, l_v + t_{v,1}\}$, and $d_v = \max\{l_v, r_v - t_{v,2}\}$. The values $\phi_{v,1} = arc\cot(c_v - a_v)$ and $\phi_{v,2} = arc\cot(b_v - d_v)$ are the* left slope *and the* right slope *of $\overline{T}_v$, respectively. Moreover, for every unbounded vertex $v \in V_U$, $\phi_v = \phi_{v,1} = \phi_{v,2}$ is the* slope *of $\overline{T}_v$.*

An example is depicted in Figure 1, where $\overline{T}_u$ and $\overline{T}_v$ correspond to bounded vertices u and v, and $\overline{T}_w$ corresponds to an unbounded vertex w. For each of these trapezoids, the corresponding interval (together with the associated tolerant points, if the vertex is bounded) is drawn above the trapezoid for better visibility. The left (resp. right) tolerant points are depicted by a square (resp. cycle). Observe that when a vertex v is bounded, the values c_v and d_v coincide with the tolerant points l_{t_v} and r_{t_v}, respectively, while $\phi_{v,1} = arc\cot t_{v,1}$ and $\phi_{v,2} = arc\cot t_{v,2}$. On the other hand, when a vertex v is unbounded, the values c_v and d_v coincide with the endpoints b_v and a_v of I_v, respectively, while $\phi_{v,1} = \phi_{v,2} = arc\cot |I_v|$. Observe also that in both cases where $t_{v,1} = t_{v,2} = |I_v|$ and $t_{v,1} = t_{v,2} = \infty$, the trapezoid $\overline{T}_v$ is reduced to a line segment (cf. $\overline{T}_v$ and $\overline{T}_w$ in Figure 1). Furthermore, similarly to the above, we can assume w.l.o.g. that all endpoints and slopes of the trapezoids are distinct, i.e. $\{a_u, b_u, c_u, d_u\} \cap \{a_v, b_v, c_v, d_v\} = \emptyset$ and $\{\phi_{u,1}, \phi_{u,2}\} \cap \{\phi_{v,1}, \phi_{v,2}\} = \emptyset$ for every $u, v \in V$ with $u \neq v$. Since $|I_v| > 0$ and $t_{v,1}, t_{v,2} > 0$ for every vertex v, it follows that $0 < \phi_{v,1} < \frac{\pi}{2}$ and $0 < \phi_{v,2} < \frac{\pi}{2}$ for all slopes $\phi_{v,1}, \phi_{v,2}$.

Definition 2.4. *Let $u \in V_B$ be a bounded vertex in a multitolerance representation and a_u, b_u, c_u, d_u be the endpoints of the trapezoid $\overline{T}_u$. Let $x \in [a_u, d_u]$ and $y \in [c_u, b_u]$ be two points on the lines L_2 and L_1, respectively, such that $x = \lambda a_u + (1 - \lambda) d_u$ and $y = \lambda c_u + (1-\lambda) b_u$ for the same value $\lambda \in [0, 1]$. Then $\phi_u(x)$ is the slope of the line segment with endpoints x and y on the lines L_2 and L_1, respectively.*

In the example of Figure 1, two points $x \in [a_u, d_u]$ and $y \in [c_u, b_u]$ are depicted on the lines L_2 and L_1, respectively, such that $x = \lambda a_u + (1 - \lambda) d_u$ and $y = \lambda c_u + (1 - \lambda) b_u$ for the same value $\lambda \in [0, 1]$. Then, the interval $[x, y]$ on the real line, with values x and y as endpoints, coincides with the tolerance-interval $I_{l_{t_u}, r_{t_u}}(1-\lambda) = [l_u + (r_{t_u} - l_u)(1-\lambda), l_{t_u} + (r_u - l_{t_u})(1-\lambda)]$ of $\mathcal{I}(I_u, l_{t_u}, r_{t_u})$ (cf. the definition of a multitolerance representation). Furthermore, for the slope $\phi_u(x)$, as defined in Definition 2.4 (cf. Figure 1), it follows that $\cot\phi_u(x) = y - x = \lambda(c_u - a_u) + (1-\lambda)(b_u - d_u)$. Therefore, since $\cot\phi_{u,1} = c_u - a_u$ and $\cot\phi_{u,2} = b_u - d_u$, the next observation follows.

Observation 1. *Let $x = \lambda a_u + (1-\lambda)d_u$ for a bounded vertex $u \in V_B$ and some value $\lambda \in [0,1]$. Then, $\cot\phi_u(x) = \lambda\cot\phi_{u,1} + (1-\lambda)\cot\phi_{u,2}$.*

Note that, in Definition 2.3, the endpoints a_v, b_v, c_v, d_v of any trapezoid $\overline{T}_v$ (on the lines L_1 and L_2) lie on the plane $z = 0$ in $\mathbb{R}^3$. Therefore, since we assumed that the distance between the lines L_1 and L_2 is one, these endpoints of $\overline{T}_v$ correspond to the points $(a_v, 0, 0)$, $(b_v, 1, 0)$, $(c_v, 1, 0)$, and $(d_v, 0, 0)$ in $\mathbb{R}^3$, respectively. For the sake of presentation, we may not distinguish in the following between these points in $\mathbb{R}^3$ and the corresponding real values a_v, b_v, c_v, d_v, whenever this slight abuse of notation does not cause any confusion.

We are ready to give the main definition of this article. For a set X of points in $\mathbb{R}^3$, denote by $H_{\text{convex}}(X)$ the *convex hull* defined by the points of X. That is, $\overline{T}_v = H_{\text{convex}}(a_v, b_v, c_v, d_v)$ for every vertex $v \in V$ by Definition 2.3, where a_v, b_v, c_v, d_v are points of the plane $z = 0$ in $\mathbb{R}^3$.

Definition 2.5. *Let $G = (V, E)$ be a multitolerance graph with a multitolerance representation $\{I_v = [a_v, b_v], \tau_v \mid v \in V\}$ and $\Delta = \max\{b_v \mid v \in V\} - \min\{a_v \mid v \in V\}$ be the greatest distance between two interval endpoints. For every vertex $v \in V$, the* trapezoepiped T_v *of v is the convex set of points in $\mathbb{R}^3$ defined as follows:*

(a) if $t_{v,1}, t_{v,2} \leq |I_v|$ (that is, v is bounded), then $T_v = H_{convex}(\overline{T}_v, a'_v, b'_v, c'_v, d'_v)$,

(b) if $t_v = t_{v,1} = t_{v,2} = \infty$ (that is, v is unbounded), then $T_v = H_{convex}(a'_v, c'_v)$,

where $a'_v = (a_v, 0, \Delta - \cot\phi_{v,1})$, $b'_v = (b_v, 1, \Delta - \cot\phi_{v,2})$, $c'_v = (c_v, 1, \Delta - \cot\phi_{v,1})$, and $d'_v = (d_v, 0, \Delta - \cot\phi_{v,2})$. The set of trapezoepipeds $\{T_v \mid v \in V\}$ is a trapezoepiped representation *of G.*

Note by the definition of Δ that $\Delta - \cot\phi_{v,1} \geq 0$ and $\Delta - \cot\phi_{v,2} \geq 0$ for every $v \in V$. Furthermore, observe that for each interval I_v, the trapezoid $\overline{T}_v$ of Definition 2.3 (see also Figure 1) coincides with the projection of the trapezoepiped T_v on the plane $z = 0$. An example of this construction is given in Figure 2. A multitolerance graph G with seven vertices $\{v_1, v_2, \ldots, v_7\}$ is depicted in Figure 2(a), while the trapezoepiped representation of G is illustrated in Figure 2(b). The set of bounded and unbounded vertices in this representation are $V_B = \{v_3, v_4, v_6, v_7\}$ and $V_U = \{v_1, v_2, v_5\}$, respectively. We illustrate the endpoints $a_{v_i}, b_{v_i}, c_{v_i}, d_{v_i}$ and $a'_{v_i}, b'_{v_i}, c'_{v_i}, d'_{v_i}$ of T_{v_i}, as well as the relationship between the interval I_{v_i} and the corresponding trapezoepiped T_{v_i} for one unbounded and one bounded vertex, cf. v_1 and v_6, respectively. Note that $a_{v_1} = d_{v_1}$, $a'_{v_1} = d'_{v_1}$, $c_{v_1} = b_{v_1}$, and $c'_{v_1} = b'_{v_1}$, since v_1 is unbounded. In the case where $t_{v_i,1}, t_{v_i,2} < |I_{v_i}|$, the trapezoepiped T_{v_i} is three-dimensional, cf. T_{v_3}, T_{v_4}, and T_{v_6}, while in the border case where $t_{v_i,1} = t_{v_i,2} = |I_{v_i}|$ it degenerates to a two-dimensional rectangle, cf. T_{v_7}. In these two cases, each T_{v_i} corresponds to a bounded vertex v_i. In the remaining case where v_i is unbounded, i.e. $t_{v_i} = t_{v_i,1} = t_{v_i,2} = \infty$, the trapezoepiped T_{v_i} degenerates to an one-dimensional line segment above plane $z = 0$, cf. T_{v_1}, T_{v_2}, and T_{v_5}.

We now prove that the trapezoepiped representation forms a 3-dimensional intersection model for the class of multitolerance graphs (namely, that every multitolerance graph G can be viewed as the intersection graph of the corresponding trapezoepipeds T_v).

Theorem 2.1. *Let $G = (V, E)$ be a multitolerance graph with a multitolerance representation $\{I_v = [a_v, b_v], \tau_v \mid v \in V\}$. Then for every $u, v \in V$, $uv \in E$ if and only if $T_u \cap T_v \neq \emptyset$.*

Clearly, for each $v \in V$ the trapezoepiped T_v can be constructed in constant time; therefore the next lemma follows directly.

Lemma 2.1. *Given a multitolerance representation of a multitolerance graph G with n vertices, a trapezoepiped representation of G can be constructed in $O(n)$ time.*

3 A canonical representation of multitolerance graphs.

In this section we introduce a canonical representation of multitolerance graphs, which is a special kind of a trapezoepiped representation. Moreover, we present an efficient algorithm that constructs in $O(n \log n)$ time a canonical representation of a multitolerance graph G with n vertices, given any trapezoepiped representation of G. This algorithm proves to be useful for designing efficient algorithms on multitolerance graphs for the minimum coloring and the maximum clique problems with optimal running time $O(n \log n)$, as we will present in Section 4. First, we state the following definition, similarly to the case of tolerance graphs [20] (see also [10, 11]).

Definition 3.1. *An unbounded vertex $v \in V_U$ of a multitolerance graph G is called* inevitable *(for a certain trapezoepiped representation), if replacing T_v by*

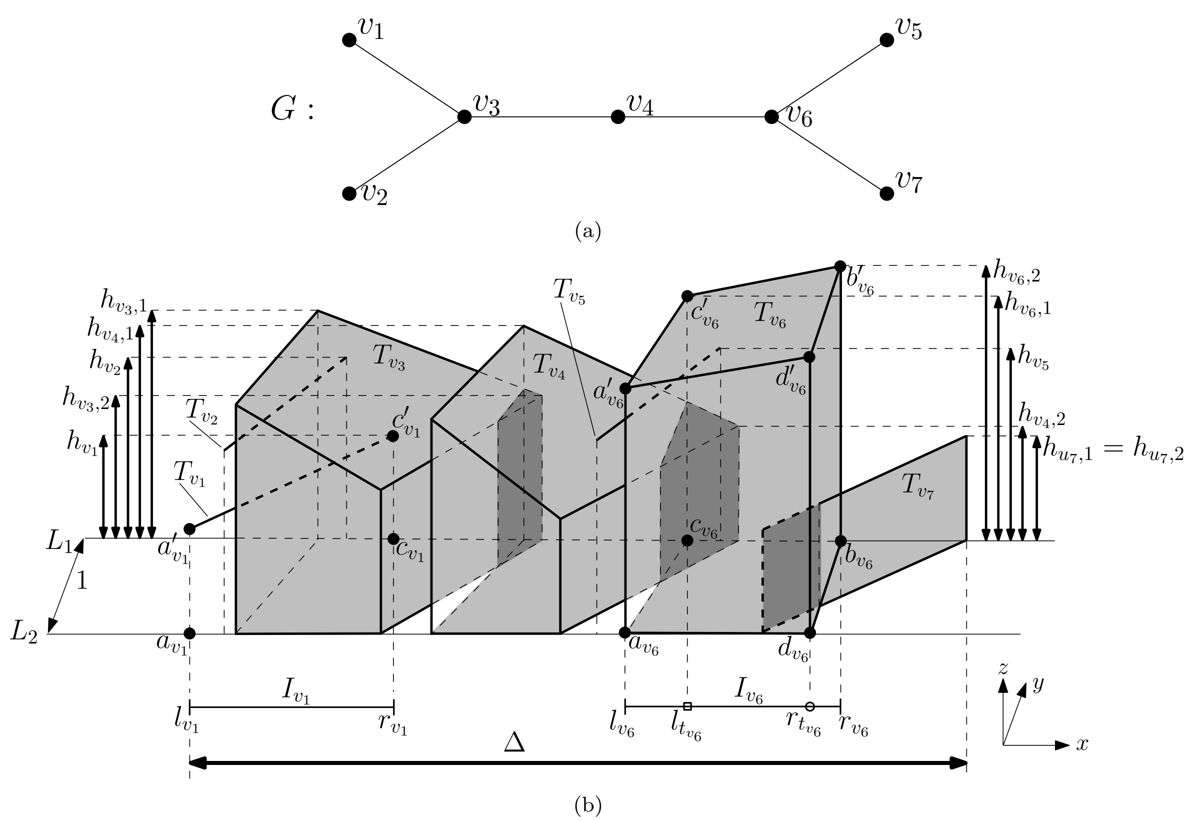

Figure 2: (a) A multitolerance graph G and (b) a trapezoepiped representation R of G. Here, $h_{v_i,j} = \Delta - \cot \phi_{v_i,j}$ for every bounded vertex $v_i \in V_B$ and $j \in \{1, 2\}$, while $h_{v_i} = \Delta - \cot \phi_{v_i}$ for every unbounded vertex $v_i \in V_U$.

$H_{convex}(\overline{T}_v, a'_v, c'_v)$ *creates a new edge* uv *in* G*; then* u *is a* hovering vertex *of* v *and the set* $H(v)$ *of all hovering vertices of* v *is the* hovering set *of* v*. Otherwise,* v *is called* evitable.

Recall that $a'_v = d'_v$ and $c'_v = b'_v$ for every unbounded vertex $v \in V_U$, and thus $H_{\text{convex}}(\overline{T}_v, a'_v, c'_v) = H_{\text{convex}}(\overline{T}_v, a'_v, b'_v, c'_v, d'_v)$ in Definition 3.1. Therefore, replacing T_v by $H_{\text{convex}}(\overline{T}_v, a'_v, c'_v)$ in the trapezoepiped representation of G is equivalent with replacing in the corresponding multitolerance representation of G the infinite tolerance $t_v = \infty$ by the finite tolerances $t_{v,1} = t_{v,2} = |I_v|$, i.e. with making v a bounded vertex. Note that the hovering set of an inevitable unbounded vertex v can have more than one elements, since the replacement of T_v by $H_{\text{convex}}(\overline{T}_v, a'_v, c'_v)$ may create more than one new edges in G. Furthermore, $uv \notin E$ for every hovering vertex u of v, while u can be both bounded or unbounded. In the next definition we introduce the notion of a canonical representation of a multitolerance graph G.

Definition 3.2. *A trapezoepiped representation of a multitolerance graph* G *is called* canonical *if every unbounded vertex is inevitable.*

For example, in the multitolerance graph depicted in Figure 2, v_2 and v_5 are inevitable unbounded vertices, v_1 and v_4 are hovering vertices of v_2 and v_5, respectively, while v_1 is an evitable unbounded vertex. Therefore, this representation is not canonical for the graph G. However, if we replace T_{v_1} by $H_{\text{convex}}(\overline{T}_{v_1}, a'_{v_1}, c'_{v_1})$, we get a canonical representation for G.

3.1 The construction of a canonical representation.

In this section we present Algorithm 1 that constructs a canonical representation of a multitolerance graph G, given a trapezoepiped representation of G. To this end, we first provide some notions of computational geometry, which play a crucial role in our algorithm.

Definition 3.3. *Let* L *be a set of line segments in the plane. The* lower envelope $Env(L)$ *of* L *is the set of those points* $p = (x, y)$ *of the line segments of* L*, such that the point* (x, y') *does not belong to any line segment of* L*, for any* $y' < y$.

An example of a set L of non-vertical line segments in the Euclidean plane is illustrated in Figure 3. In this figure, the lower envelope $Env(L)$ of L is drawn gray for better visibility.

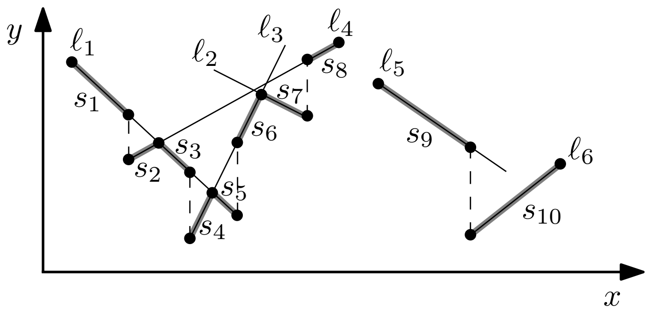

Figure 3: A set $L = \{\ell_1, \ldots, \ell_6\}$ of line segments in the plane and the lower envelope $H_{\text{lower}}(L)$ of L, which consists of the line segments $\{s_1, \ldots, s_{10}\}$.

The lower envelope $Env(L)$ of such a set L consists also of line segments (cf. Figure 3), and thus $Env(L)$ can be also specified by the endpoints of its segments. Given a set of n line segments in the plane, the lower envelope of these segments can be computed in $O(n \log n)$ time using the algorithm presented in [13]. During the computation of $Env(L)$ by this algorithm, we can in the same time also store for every line segment s of $Env(L)$ the line segment ℓ of L, in which s belongs.

We define now two subsets U_1 and U_2 of the set of inevitable unbounded vertices.

Definition 3.4. *Let $v \in V_U$ be an inevitable unbounded vertex. Then, $v \in U_1$ (resp. $v \in U_2$) if there exists at least one hovering vertex $u \in H(v)$ of v, such that u is unbounded (resp. u is bounded).*

Note that, given a trapezoepiped representation of a multitolerance graph G, the sets U_1 and U_2 are not necessarily disjoint, since an unbounded vertex may have both unbounded and bounded hovering vertices. On the other hand, since every inevitable unbounded vertex has at least one hovering vertex (cf. Definition 3.1), $U_1 \cup U_2$ coincides with the set of inevitable unbounded vertices.

We associate now with every unbounded vertex $v \in V_U$ the point $p_v = (x_v, y_v)$ in the Euclidean plane, where $x_v = a_v$ and $y_v = \Delta - \cot \phi_v$. Moreover, we associate with every bounded vertex $u \in V_B$ three points $p_{u,1} = (x_{u,1}, y_{u,1}) = (a_u, \Delta - \cot \phi_{u,1})$, $p_{u,2} = (x_{u,2}, y_{u,2}) = (d_u, \Delta - \cot \phi_{u,2})$, and $p_{u,3} = (x_{u,3}, y_{u,3}) = (b_u, \Delta)$ in the plane. Furthermore, we associate with every bounded vertex $u \in V_B$ two line segments $\ell_{u,1} = (p_{u,1}, p_{u,2})$ and $\ell_{u,2} = (p_{u,2}, p_{u,3})$ in the plane, which have the points $p_{u,1}, p_{u,2}$ and $p_{u,2}, p_{u,3}$ as endpoints, respectively.

An example of this construction is given in Figure 4, where the points p_{v_1} and p_{v_2} are associated with the unbounded vertices v_1 and v_2, respectively, while the points $p_{u,1}, p_{u,2}, p_{u,3}$ and the line segments $\ell_{u,1}, \ell_{u,2}$ are associated with the bounded vertex u.

Algorithm 1 Construction of a canonical representation of a multitolerance graph G

Input: A trapezoepiped representation R of a given multitolerance graph $G = (V, E)$

Output: A canonical representation R' of G and a hovering vertex u for every inevitable unbounded vertex v of G

```
L ← ∅; R' ← R; R'' ← R \ {T_v | v ∈ V_B}
for every vertex v ∈ V do
    if v ∈ V_U then
        p_v ← (a_v, Δ − cot φ_v)
    else {v ∈ V_B}
        p_{v,1} ← (a_v, Δ − cot φ_{v,1});
        p_{v,2} ← (d_v, Δ − cot φ_{v,2}); p_{v,3} ← (b_v, Δ)
        ℓ_{v,1} = (p_{v,1}, p_{v,2}); ℓ_{v,2} = (p_{v,2}, p_{v,3});
        L ← L ∪ {ℓ_{v,1}, ℓ_{v,2}}
Compute the set U_1 of inevitable unbounded vertices in R'' and a hovering vertex u ∈ V_U of v, for
    every v ∈ U_1, by the algorithm of [20]
Compute the lower envelope Env(L) of L by the algorithm of [13]
    {During the computation of Env(L), store for every line segment s of Env(L), the line segment ℓ_{u,1}
    or ℓ_{u,2} of L, in which s belongs}
for every vertex v ∈ V_u \ U_1 do
    if v lies above a segment s of Env(L) then {v ∈ U_2 \ U_1}
        Let ℓ_{u,1} or ℓ_{u,2} be the line segment of L, in which s belongs
        u ∈ V_B is a hovering vertex of v
    else {v is evitable unbounded}
        Replace T_v by H_convex(T̄_v, a'_v, c'_v) in R'
        {v is made bounded}
return R'
```

In the following, let $L = \{\ell_{u,1}, \ell_{u,2} \mid u \in V_B\}$ be the $2|V_B|$ line segments that are associated with the bounded vertices $u \in V_B$. For an arbitrary point $p = (x, y)$ in the plane, we say that p *lies above* $Env(L)$ (resp. above the line segment $\ell_{u,1}$ or $\ell_{u,2}$ of L) if there exists a point $p' = (x, y')$ of $Env(L)$ (resp. of $\ell_{u,1}$ or $\ell_{u,2}$), such that $y > y'$. The next lemma, which is crucial for the analysis of Algorithm 1, characterizes the vertices of U_2 using the lower envelope $Env(L)$ of L.

Lemma 3.1. *Let $v \in V_U$ be an unbounded vertex and $p_v = (x_v, y_v)$ be the associated point in the plane. Then, $v \in U_2$ if and only if p_v lies above $Env(L)$. Furthermore,* *if p_v lies above the segment $\ell_{u,1}$ or $\ell_{u,2}$ of L, then the bounded vertex u is a hovering vertex of v.*

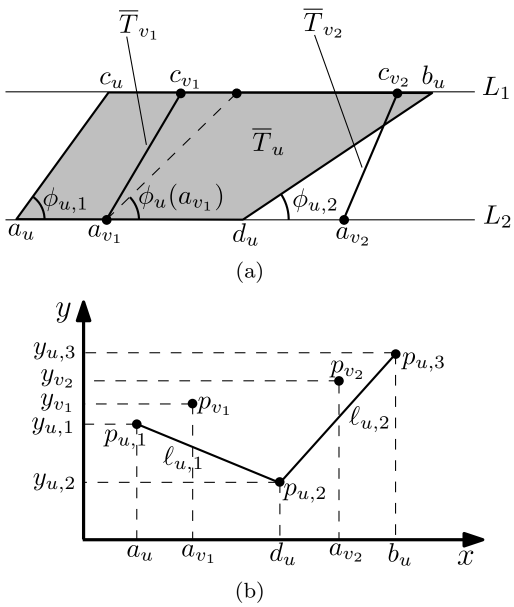

Figure 4: Two inevitable unbounded vertices $v_1, v_2 \in U_2$, where the bounded vertex $u \in V_B$ is a hovering vertex of both v_1 and v_2: (a) the trapezoids $\overline{T}_u, \overline{T}_{v_1}, \overline{T}_{v_2}$ and (b) the line segments $\ell_{u,1}, \ell_{u,2}$ of vertex u and the points p_{v_1}, p_{v_2} of vertices v_1, v_2, respectively, where $y_{u,1} = \Delta - \cot(\phi_{u,1})$, $y_{u,2} = \Delta - \cot(\phi_{u,2})$, $y_{u,3} = \Delta$, and $y_{v,j} = \Delta - \cot(\phi_{u,j})$ for every $j = 1, 2$.

The next theorem shows that, given a trapezoepiped representation, we can construct by Algorithm 1 a canonical representation in $O(n \log n)$ time. This result plays a central role in the time complexity analysis of the algorithms of Section 4.

THEOREM 3.1. *Every trapezoepiped representation of a multitolerance graph G with n vertices can be transformed by Algorithm 1 to a canonical representation of G in $O(n \log n)$ time.*

4 Coloring and clique Algorithms in $O(n \log n)$ time.

In this section we present optimal $O(n \log n)$ time algorithms for the minimum coloring and the maximum clique problems on a multitolerance graph G with n vertices, given any trapezoepiped representation of G. These algorithms mainly use Algorithm 1 to compute efficiently a canonical representation of G, as well as the coloring and clique algorithms for trapezoid graphs in [6], respectively.

Algorithm 2 Computation of a minimum coloring of a multitolerance graph G

Input: A trapezoepiped representation R of a given multitolerance graph $G = (V, E)$
Output: A minimum coloring of G

Construct a canonical representation R' of G by Algorithm 1, where a hovering vertex u_v is associated with every inevitable unbounded vertex v
Let V_B and V_U be the bounded and (inevitable) unbounded vertices of G in R', respectively
Color $G[V_B]$ by the algorithm of [6]
for every vertex $v \in V_U$ **do**
  Create a pointer from the hovering vertex u_v of v to the vertex v
for every vertex $u \in V_B$ that has at least one pointer **do**
  Assign the color of u to every vertex $v \in V_U$ that is reachable from u by a sequence of pointers

THEOREM 4.1. *A minimum coloring of a multitolerance graph G with n vertices can be computed by Algorithm 2 in optimal $O(n \log n)$ time.*

THEOREM 4.2. *A maximum clique of a multitolerance graph G with n vertices can be computed in optimal $O(n \log n)$ time.*

Proof. First we compute a canonical representation of G in $O(n \log n)$ time by Algorithm 1. By the correctness of Algorithm 2, cf. the proof of Theorem 4.1, it follows that $\chi(G) = \chi(G[V_B])$, where $\chi(H)$ denotes the chromatic number of a given graph H. Since multitolerance graphs are perfect graphs [22], $\omega(G) = \chi(G)$ and $\omega(G[V_B]) = \chi(G[V_B])$, where $\omega(H)$ denotes the clique number of a given graph H. Therefore $\omega(G) = \omega(G[V_B])$. We compute now a maximum clique Q of the bounded multitolerance (i.e. trapezoid) graph $G[V_B]$ in $O(n \log n)$ time by the algorithm presented in [6] for trapezoid graphs. Then, since $\omega(G) = \omega(G[V_B])$, Q is a maximum clique of G as well. Finally, since $\Omega(n \log n)$ is a lower bound for the time complexity of the maximum clique problem on tolerance graphs [20] and on trapezoid graphs [6], it follows that the clique algorithm for multitolerance graphs has also optimal running time.

5 Weighted independent set algorithm in $O(m + n \log n)$ time.

In this section we present Algorithm 3 that computes the value of a maximum weight independent set of a multitolerance graph $G = (V, E)$ with n vertices and m edges in $O(m + n \log n)$ time, given a trapezoepiped representation of G and a weight $w(v) > 0$ for every $v \in V$. A slight modification of this algorithm computes in the same time also a maximum weight independent set of G, instead of its value. Although the algorithm presented in [20] for the maximum weight independent set on tolerance graphs with complexity $O(n^2)$ can be extended with the same time complexity to the case of multitolerance graphs with a given trapezoepiped representation, we present here a new algorithm for multitolerance graphs that achieves a better running time $O(m + n \log n)$. Thus this algorithm improves also the best known running time of $O(n^2)$ for the maximum weight independent set on tolerance graphs [20]. Note here that $\Omega(n \log n)$ is a lower bound for the time complexity of this problem on trapezoid graphs [6], and thus also on multitolerance graphs.

First, given a trapezoepiped representation of a multitolerance graph $G = (V, E)$, we sort on the line L_2 the points $\{a_v, d_v \mid v \in V\}$ of the trapezoids $\overline{T}_v$, $v \in V$, and we visit these points sequentially from right to left. Note that $a_v = d_v$ for every unbounded vertex $v \in V_U$. A vertex v is called *processed* only after we visit the endpoint a_v of $\overline{T}_v$. During the execution of the algorithm we maintain two finite sets M and H of $O(n)$ weighted markers each on the line L_1, which are placed at some points c_v, where $v \in V$. We maintain the sets M and H in such a way that values can be inserted to and deleted from these sets, as well as the predecessor or successor of a given query value can be found. Using binary search trees, for instance AVL-trees, all these operations can be executed in $O(\log n)$ time [12]. In the following of the analysis of Algorithm 3, we will use for simplicity of the presentation the variable m to denote a marker of the set M (rather than the number of edges of G); furthermore, we will refer by $|E|$ to the number of edges of G. For every marker $m \in M$ (resp. $h \in H$), we denote by p_m (resp. p_h) the point of L_1, at which the marker m (resp. h) is placed.

The markers of the set M are placed at points c_v on the line L_1, for some bounded vertices $v \in V_B$. After an iteration of the algorithm, where the vertices of the set $U \subseteq V$ have been processed, the weight $W(m)$ of a marker m placed at the point c_v on the line L_1 equals the maximum weight of an independent set, which includes only vertices $u \in U$ such that $c_v \leq c_u$. Moreover, a marker m is placed at c_v only if such a maximum weight independent set includes the (bounded) vertex v.

The markers of the set H are placed at points c_v on the line L_1, where $v \in V_U$. After an iteration of the algorithm, where the vertices of the set $U \subseteq V$ have been processed, there is a weighted marker $h \in H$ placed at the point c_v on L_1, for every unbounded vertex $v \in V_u \cap U$, while the weight $w(h)$ of h equals the weight $w(v)$ of vertex v. Furthermore, in the AVL-tree of the set H, we store at every internal vertex x also a label with the total weight of the tree that consists of x and its right subtree. Note that

Algorithm 3 Maximum weight independent set of a multitolerance graph G

Input: A trapezoepiped representation of a given multitolerance graph $G = (V, E)$
Output: The value of a maximum weight independent set of G

Place a marker m_0 at the point $p_{m_0} = \max\{b_v \mid v \in V\} + 1$ of the line L_1
$W(m_0) \leftarrow 0$; $M \leftarrow \{m_0\}$
for every $v \in V_B$ **do** {initialization}
  $W(v) \leftarrow 0$
  Compute the value $\widetilde{w}_v = \sum\{w(u) \mid u \in V_U, c_u \in (c_v, b_v)\}$
  for every $u \in N(v)$ **do**
    if $u \in V_U$ and $c_u \in (c_v, b_v)$ **then** {v is not a hovering vertex of u}
      $\widetilde{w}_v \leftarrow \widetilde{w}_v - w(u)$
for every point $p \in \{a_v, d_v \mid v \in V\}$ from right to left **do** {p lies on the line L_2}
  if $p = a_v$ for some $v \in V_U$ **then** {the unbounded vertex v is being processed}
    Insert a new marker $h \in H$ at the point $p_h = c_v$
    $w(h) \leftarrow w(v)$
    $m \leftarrow$ the leftmost marker of M to the right of c_v on L_1
    Remove all markers $m' \in M$ to the left of m, for which $W(m') \leq W(m) + w(H[p_{m'}, p_m))$
  if $p = d_v$ for some $v \in V_B$ **then**
    $m \leftarrow$ the leftmost marker of M to the right of b_v on L_1
    $W(v) \leftarrow (w(v) + \widetilde{w}_v) + W(m) + w(H[b_v, p_m))$ {do not modify the markers of M}
  if $p = a_v$ for some $v \in V_B$ **then** {the bounded vertex v is being processed}
    $m \leftarrow$ the leftmost marker of M to the right of c_v on L_1
    if $W(v) > W(m) + w(H[c_v, p_m))$ **then**
      Insert a new marker $m' \in M$ at the point $p_{m'} = c_v$
      $W(m') \leftarrow W(v)$
      Remove all markers $m'' \in M$ to the left m', for which $W(m'') \leq W(m') + w(H[p_{m''}, p_{m'}))$
return $W(m) + w(H[c, p_m))$, where $c = \min\{c_v \mid v \in V\} - 1$ and m is the leftmost marker of M

after an insertion of a new marker h to the AVL-tree that stores H, we can update in $O(\log n)$ time these labels of the internal vertices, as follows. First, we need to update a constant number of labels during the "trinode restructure" operation (for more details, see [12]). Then, following the path from the interval vertex that stores the new marker h to the root, we add the weight $w(h)$ of h to the label of every internal vertex that has h in its right subtree.

For every two points q and q' on L_1, where $q < q'$, denote for simplicity by $H[q, q')$ (resp. $H[q, +\infty)$) the set of the markers h in the current set H that have been placed in the semi-closed interval $[q, q')$ (resp. in the subline $[q, +\infty)$) of L_1. Denote also by $w(H[q, q'))$ (resp. $w(H[q, +\infty))$) the sum of the weights of the markers $h \in H[q, q')$ (resp. $h \in H[q, +\infty)$). For simplicity, in the case where $q' = q$, we set $w(H[q, q)) = 0$. Furthermore, note that if $q \leq q' \leq q''$, then $w(H[q, q'')) = w(H[q, q')) + w(H[q', q''))$. For every point q on L_1, we can compute in $O(\log n)$ time the value $w(H[q, +\infty))$, as follows. First, we locate in $O(\log n)$ time the leftmost marker $h \in H$ that has been placed at a point q', such that $q \leq q'$. Then, we follow in the AVL-tree of H the path from the root to the internal vertex x that stores h and sum up the label stored at x and the labels stored at the internal vertices of this path, at which we follow the left child. Furthermore, since $w(H[q, q')) = w(H[q, +\infty)) - w(H[q', +\infty))$ for every two points q, q' on the line L_1 such that $q \leq q'$, we can compute the value $w(H[q, q'))$ in $O(\log n)$ time as well.

The correctness and the running time of Algorithm 3 are provided by the next theorem.

THEOREM 5.1. *A maximum weight independent set of a multitolerance graph $G = (V, E)$ with n vertices can be computed using Algorithm 3 in $O(|E| + n \log n)$ time.*

5.1 An optimal $O(n \log n)$ time algorithm for tolerance graphs. In this section we prove that if the input graph $G = (V, E)$ is a tolerance graph with n vertices, we can slightly modify Algorithm 3, such that it computes a maximum weight independent set of G in optimal $O(n \log n)$ time. In particular, if G is a tolerance graph, the trapezoepiped T_v of every bounded vertex $v \in V_B$ in the trapezoepiped representation of G reduces to a parallelepiped, since in this case $\phi_{v,1} = \phi_{v,2}$. Using this property of the trapezoepiped representation of tolerance graphs, we manage to compute the values $\widetilde{w}_v$ for all $v \in V_B$ in $O(n \log n)$ time, instead of $O(|E| + n \log n)$ time in lines 3-8 of Algorithm 3. Therefore, since the execution of all the remaining lines of Algorithm 3 (except lines 3-8) can be done in $O(n \log n)$ time, the next theorem follows.

THEOREM 5.2. *A maximum weight independent set of a tolerance graph $G = (V, E)$ with n vertices edges can be computed in $O(n \log n)$ time, which is optimal.*

6 Classification of multitolerance graphs.

In this section we classify the class of multitolerance graphs inside the hierarchy of perfect graphs given in [11] (in Figure 2.8). The resulting hierarchy of classes of perfect graphs is *complete*, i.e. all inclusions are strict[1]. This hierarchy is presented in Figure 5. We prove these results by using the trapezoepiped representation of multitolerance graphs presented in Section 2, as well as some known results on the hierarchy of perfect graphs given in [11].

First we briefly review the classes shown in Figure 5. A graph is *perfect* if the chromatic number of every induced subgraph equals the clique number of this subgraph. A graph G is called *alternately orientable* if there exists an orientation F of G which is transitive on every chordless cycle of length at least 4, i.e. the directions of the oriented edges must alternate. A graph G is called *weakly chordal* (or *weakly triangulated*) if G has no induced subgraph isomorphic to the chordless cycle C_n with n vertices, or to its complement $\overline{C_n}$, for any $n \geq 5$. A vertex order $\prec$ of a graph G is called *perfect* if and only if G contains no induced path $abcd$ with $a \prec b$ and $d \prec c$. A graph G is called *co-perfectly orderable* if its complement $\overline{G}$ admits a perfect order. Moreover, a *comparability* graph is a graph which can be transitively oriented and a *cocomparability* graph is a graph whose complement is a comparability graph. For more definitions we refer to [11].

We can summarize the results of this section in the following theorem.

THEOREM 6.1. *Multitolerance graphs:*

(a) strictly *include tolerance and trapezoid graphs,*

(b) are strictly *included in weakly chordal graphs and in co-perfectly orderable graphs,*

(c) are incomparable *with alternately orientable and with cocomparability graphs.*

[1]It was claimed in [22] (in Theorem 3.1(b)) that tolerance graphs are *strictly* included in multitolerance graphs; however, in the proof of that theorem only inclusion has been shown, and not strict inclusion. We prove strict inclusion in Theorem 6.1(a). Moreover, it has been correctly shown in [22] that a multitolerance graph does not contain any chordless cycle C_n, where $n \geq 5$. We prove in Theorem 6.1(b) that actually the same holds also for the complements $\overline{C}_n$ of C_n, where $n \geq 5$, and thus every multitolerance graph is weakly chordal.

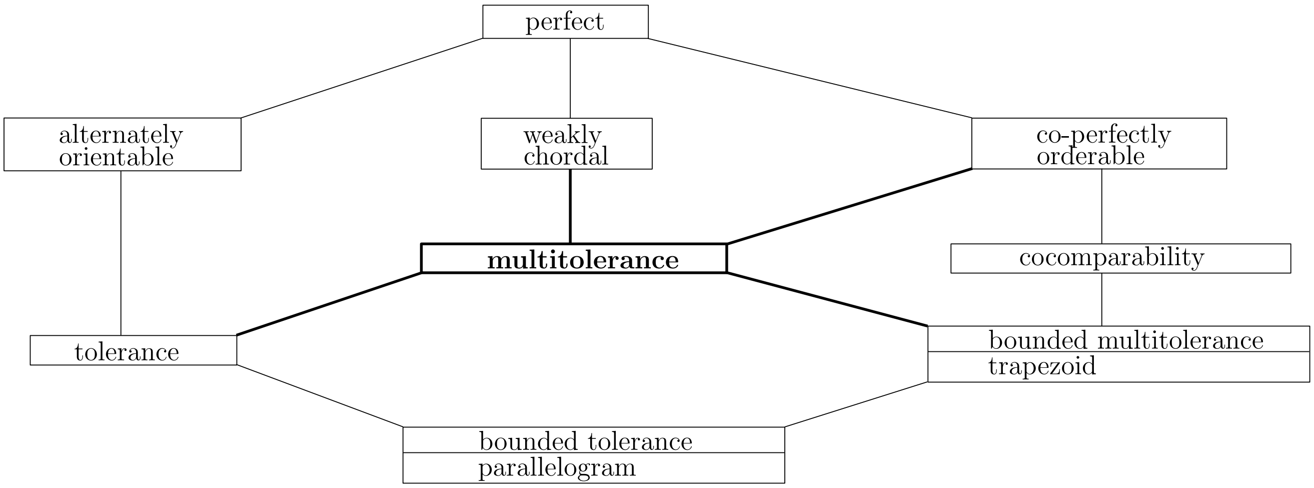

Figure 5: The classification of multitolerance graphs in the hierarchy of perfect graphs in [11]. This hierarchy is complete, i.e. every inclusion is strict.

7 Conclusions and further research.

In this article we proposed the first non-trivial intersection model for general multitolerance graphs, given by objects in the 3-dimensional space, called trapezoepipeds. This trapezoepiped representation unifies in a simple and intuitive way the well known trapezoid representation for bounded multitolerance graphs and the recently introduced parallelepiped representation for tolerance graphs in [20]. Using this representation, we presented efficient algorithms that compute a minimum coloring, a maximum clique, and a maximum weight independent set on a multitolerance graph, respectively. The running times of the first two algorithms are optimal, while the third algorithm improves the best known running time for the maximum weight independent set on tolerance graphs. In particular, a variation of the latter algorithm computes a maximum weight independent set of a tolerance graph in optimal time, closing thus the complexity gap of [20]. Furthermore, we proved several structural results on the class of multitolerance graphs, which complement the hierarchy of perfect graphs given in [11]. The proposed intersection model provides geometric insight for multitolerance graphs and it can be expected to prove useful in deriving new algorithmic and structural results. The recognition problem for general multitolerance graphs, remains an interesting open problem. On the contrary, it is known that trapezoid (i.e. bounded multitolerance) graphs can be recognized efficiently [17,19], while it is NP-complete to recognize tolerance and bounded tolerance graphs [21], as well as max-tolerance graphs [15].

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