On the Intersection of Tolerance and Cocomparability Graphs

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Abstract. It has been conjectured by Golumbic and Monma in 1984 that the intersection of tolerance and cocomparability graphs coincides with bounded tolerance graphs. Since cocomparability graphs can be efficiently recognized, a positive answer to this conjecture in the general case would enable us to efficiently distinguish between tolerance and bounded tolerance graphs, although it is NP-complete to recognize each of these classes of graphs separately. The conjecture has been proved under some – rather strong – *structural* assumptions on the input graph; in particular, it has been proved for complements of trees, and later extended to complements of bipartite graphs, and these are the only known results so far. Furthermore, it is known that the intersection of tolerance and cocomparability graphs is contained in the class of trapezoid graphs. In this article we prove that the above conjecture is true for every graph G, whose tolerance representation satisfies a slight assumption; note here that this assumption concerns only the given tolerance *representation* R of G, rather than *any* structural property of G. This assumption on the representation is guaranteed by a wide variety of graph classes; for example, our results immediately imply the correctness of the conjecture for complements of triangle-free graphs (which also implies the above-mentioned correctness for complements of bipartite graphs). Our proofs are algorithmic, in the sense that, given a tolerance representation R of a graph G, we describe an algorithm to transform R into a bounded tolerance representation R^* of G. Furthermore, we conjecture that any minimal tolerance graph G that is not a bounded tolerance graph, has a tolerance representation with exactly one unbounded vertex. Our results imply the non-trivial result that, in order to prove the conjecture of Golumbic and Monma, it suffices to prove our conjecture. In addition, there already exists evidence in the literature that our conjecture is true.

Keywords: Tolerance graphs, cocomparability graphs, 3-dimensional intersection model, trapezoid graphs, parallelogram graphs.

1 Introduction

A simple undirected graph $G = (V, E)$ on n vertices is called a *tolerance* graph if there exists a collection $I = \{I_u \mid u \in V\}$ of closed intervals on the real line

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and a set $t = \{t_u \mid u \in V\}$ of positive numbers, such that for any two vertices $u, v \in V$, $uv \in E$ if and only if $|I_u \cap I_v| \geq \min\{t_u, t_v\}$. The pair $\langle I, t\rangle$ is called a *tolerance representation* of G. A vertex u of G is called a *bounded vertex* (in a certain tolerance representation $\langle I, t\rangle$ of G) if $t_u \leq |I_u|$; otherwise, u is called an *unbounded vertex* of G. If G has a tolerance representation $\langle I, t\rangle$ where all vertices are bounded, then G is called a *bounded tolerance* graph and $\langle I, t\rangle$ a *bounded tolerance representation* of G.

Tolerance graphs find numerous applications (in bioinformatics, constrained-based temporal reasoning, resource allocation, and scheduling problems, among others). Since their introduction in 1982 [9], these graphs have attracted many research efforts [2, 4, 7, 10–12, 15, 16], as they generalize in a natural way both interval and permutation graphs [9]; see [12] for a detailed survey.

Given an undirected graph $G = (V, E)$ and a vertex subset $M \subseteq V$, M is called a *module* in G, if for every $u, v \in M$ and every $x \in V \setminus M$, x is either adjacent in G to both u and v or to none of them. Note that $\emptyset$, V, and all singletons $\{v\}$, where $v \in V$, are trivial modules in G. A *comparability* graph is a graph which can be transitively oriented. A *cocomparability* graph is a graph whose complement is a comparability graph. A *trapezoid* (resp. *parallelogram* and *permutation*) graph is the intersection graph of trapezoids (resp. parallelograms and line segments) between two parallel lines L_1 and L_2 [8]. Such a representation with trapezoids (resp. parallelograms and line segments) is called a *trapezoid* (resp. *parallelogram* and *permutation*) *representation* of this graph. A graph is bounded tolerance if and only if it is a parallelogram graph [2]. Permutation graphs are a strict subset of parallelogram graphs [3]. Furthermore, parallelogram graphs are a strict subset of trapezoid graphs [19], and both are subsets of cocomparability graphs [8, 12]. On the other hand, not every tolerance graph is a cocomparability graph [8, 12].

Cocomparability graphs have received considerable attention in the literature, mainly due to their interesting structure that leads to efficient algorithms for several NP-hard problems, see e.g. [5, 6, 12, 14]. Furthermore, the intersection of the class of cocomparability graphs with other graph classes has interesting properties and coincides with other widely known graph classes. For instance, their intersection with chordal graphs is the class of interval graphs, while their intersection with comparability graphs is the class of permutation graphs [8]. These structural characterizations find also direct algorithmic implications to the recognition problem of interval and permutation graphs, respectively, since the class of cocomparability graphs can be recognized efficiently [8, 20]. In this context, the following conjecture has been made in 1984 [10]:

Conjecture 1 ([10]). The intersection of cocomparability graphs with tolerance graphs is exactly the class of bounded tolerance graphs.

Note that the inclusion in one direction is immediate: every bounded tolerance graph is a cocomparability graph [8, 12], as well as a tolerance graph by definition. Conjecture 1 has been proved for complements of trees [1], and later extended to complements of bipartite graphs [18], and these are the only known results so far. Furthermore, it has been proved that the intersection of tolerance and cocomparability graphs is contained in the class of trapezoid graphs [7].

Since cocomparability graphs can be efficiently recognized [20], a positive answer to Conjecture 1 would enable us to efficiently distinguish between tolerance and bounded tolerance graphs, although it is NP-complete to recognize both tolerance and bounded tolerance graphs [16]. Recently, an intersection model for general tolerance graphs has been presented in [15], given by 3D-parallelepipeds. This *parallelepiped representation* of tolerance graphs generalizes the parallelogram representation of bounded tolerance graphs; the main idea is to exploit the third dimension to capture the information given by unbounded tolerances.

Our contribution. In this article we prove that Conjecture 1 is true for every graph G, whose parallelepiped representation R satisfies a slight assumption (to be defined later). This assumption is guaranteed by a wide variety of graph classes; for example, our results immediately imply correctness of the conjecture for complements of triangle-free graphs (which also implies the above mentioned correctness for complements of trees [1] and complements of bipartite graphs [18]). Furthermore, we state a new conjecture regarding only the separating examples between tolerance and bounded tolerance graphs (cf. Conjecture 2). There already exists evidence in the literature that this conjecture is true [12]. Our results reduce Conjecture 1 to our conjecture; that is, the correctness of our conjecture implies the correctness of Conjecture 1.

Specifically, we state three conditions on the unbounded vertices of G (in the parallelepiped representation R). Condition 1 is that R has exactly one unbounded vertex. Condition 2 is that, for every unbounded vertex u of G (in R), there exists no unbounded vertex v of G whose neighborhood is strictly included in the neighborhood of u. Note that these two conditions concern only the parallelepiped representation R; furthermore, the second condition is weaker than the first one. Then, Condition 3 concerns also the position of the unbounded vertices in the trapezoid representation R_T, and it is weaker than the other two.

Assuming that this (weaker) Condition 3 holds, we algorithmically construct a parallelogram representation of G, thus proving that G is a bounded tolerance graph. The proof of correctness relies on the fact that G can be represented simultaneously by R and by R_T. The main idea is to iteratively "eliminate" the unbounded vertices of R. That is, assuming that the input representation R has $k \geq 1$ unbounded vertices, we choose an unbounded vertex u in R and construct a parallelepiped representation R^* of G with $k-1$ unbounded vertices; specifically, R^* has the same unbounded vertices as R except for u (which becomes bounded in R^*). As a milestone in the above construction of the representation R^*, we construct an induced subgraph G_0 of G that includes u, with the property that the vertex set of $G_0 \setminus \{u\}$ is a module in $G \setminus \{u\}$. The presented techniques are new and provide geometrical insight for the graphs that are both tolerance and cocomparability.

In order to state our Conjecture 2, we define a graph G to be a *minimally unbounded tolerance* graph, if G is tolerance but not bounded tolerance, while G becomes bounded tolerance if we remove any vertex of G.

Conjecture 2. Any minimally unbounded tolerance graph has a tolerance representation with exactly one unbounded vertex.

In other words, Conjecture 2 states that any minimally unbounded tolerance graph G has a tolerance representation (or equivalently, a parallelepiped representation) R that satisfies Condition 1 (stated above). Our results imply the non-trivial result that, in order to prove Conjecture 1, it suffices to prove Conjecture 2. In addition, there already exists evidence that Conjecture 2 is true, as to the best of our knowledge it is true for all known examples of minimally unbounded tolerance graphs in the literature (see e.g. [12]).

Organization of the paper. We first review in Section 2 some properties of tolerance and trapezoid graphs. Then we define the notion of a *projection representation* of a tolerance graph G, which is an alternative way to think about a parallelepiped representation of G. Furthermore, we introduce the *right* and *left border properties* of a vertex in a projection representation, which are crucial for our analysis. In Section 3 we prove our main results and we discuss how these results reduce Conjecture 1 to Conjecture 2. Finally, we discuss the presented results and further research in Section 4. Due to space limitations, the proofs are omitted here; a full version of the paper can be found in [17].

2 Definitions and Basic Properties

Notation. We consider in this article simple undirected graphs with no loops or multiple edges. In a graph $G = (V, E)$, the edge between vertices u and v is denoted by uv, and in this case u and v are called *adjacent* in G. Given a vertex subset $S \subseteq V$, $G[S]$ denotes the induced subgraph of G on the vertices in S. Whenever it is clear from the context, we may not distinguish between a vertex set S and the induced subgraph $G[S]$ of G. Furthermore, we denote for simplicity the induced subgraph $G[V \setminus S]$ by $G \setminus S$. Denote by $N(u) = \{v \in V \mid uv \in E\}$ the set of neighbors of a vertex u in G, and $N[u] = N(u) \cup \{u\}$. For any two sets A, B, we write $A \subseteq B$ if A is included in B, and $A \subset B$ if A is strictly included in B.

Consider a trapezoid graph $G = (V, E)$ and a trapezoid representation R_T of G, where vertex $u \in V$ corresponds to the trapezoid T_u in R_T. Since trapezoid graphs are also cocomparability graphs [8], we can define the partial order $(V, \ll_{R_T})$, such that $u \ll_{R_T} v$, or equivalently $T_u \ll_{R_T} T_v$, if and only if T_u lies completely to the left of T_v in R_T (and thus also $uv \notin E$). Note that there are several trapezoid representations of a particular trapezoid graph G. Given one such representation R_T, we can obtain another one R'_T by *vertical axis flipping* of R_T, i.e. R'_T is the mirror image of R_T along an imaginary line perpendicular to L_1 and L_2.

Let us now briefly review the parallelepiped representation model of tolerance graphs [15]. Consider a tolerance graph $G = (V, E)$ and let V_B and V_U denote the set of bounded and unbounded vertices of G (for a certain tolerance representation), respectively. Consider now two parallel lines L_1 and L_2 in the plane. For every vertex $u \in V$, consider a parallelogram $\overline{P}_u$ with two of its lines on L_1 and L_2, respectively, and ϕ_u be the (common) slope of the other two lines of $\overline{P}_u$ with L_1 and L_2. For every unbounded vertex $u \in V_U$, the parallelogram $\overline{P}_u$

is trivial, i.e. a line. In the model of [15], every bounded vertex $u \in V_B$ corresponds to the parallelepiped $P_u = \{(x, y, z) \mid (x, y) \in \overline{P}_u, 0 \leq z \leq \phi_u\}$ in the 3-dimensional space, while every unbounded vertex $u \in V_U$ corresponds to the line $P_u = \{(x, y, z) \mid (x, y) \in \overline{P}_u, z = \phi_u\}$. The resulting set $\{P_u \mid u \in V\}$ of parallelepipeds in the 3-dimensional space constitutes the *parallelepiped representation* of G. In this model, two vertices u, v are adjacent if and only if $P_u \cap P_v \neq \emptyset$. That is, R is an intersection model for G. For more details we refer to [15]. An example of a parallelepiped representation R is illustrated in Figure 1(a). This representation corresponds to the induced path $P_4 = (z, u, v, w)$ with four vertices (P_4 is a tolerance graph); in particular, vertex w is unbounded in R, while the vertices z, u, v are bounded in R.

Definition 1 ([15]). *An unbounded vertex $v \in V_U$ of a tolerance graph G is called* inevitable *(in a certain parallelepiped representation R), if making v a bounded vertex in R, i.e. if replacing P_v with $\{(x, y, z) \mid (x, y) \in P_v, 0 \leq z \leq \phi_v\}$, creates a new edge in G.*

Definition 2 ([15]). *A parallelepiped representation R of a tolerance graph G is called* canonical *if every unbounded vertex in R is inevitable.*

For example, the parallelepiped representation of Figure 1(a) is canonical, since w is the only unbounded vertex and it is inevitable. A canonical representation of a tolerance graph G always exists, and can be computed in $O(n \log n)$ time, given a parallelepiped representation of G, where n is the number of vertices of G [15].

Given a parallelepiped representation R of the tolerance graph G, we define now an alternative representation, as follows. Let $\overline{P}_u$ be the projection of P_u to the plane $z = 0$ for every $u \in V$. Then, for two bounded vertices u and v, $uv \in E$ if and only if $\overline{P}_u \cap \overline{P}_v \neq \emptyset$. Furthermore, for a bounded vertex v and an unbounded vertex u, $uv \in E$ if and only if $\overline{P}_u \cap \overline{P}_v \neq \emptyset$ and $\phi_v > \phi_u$. Moreover,

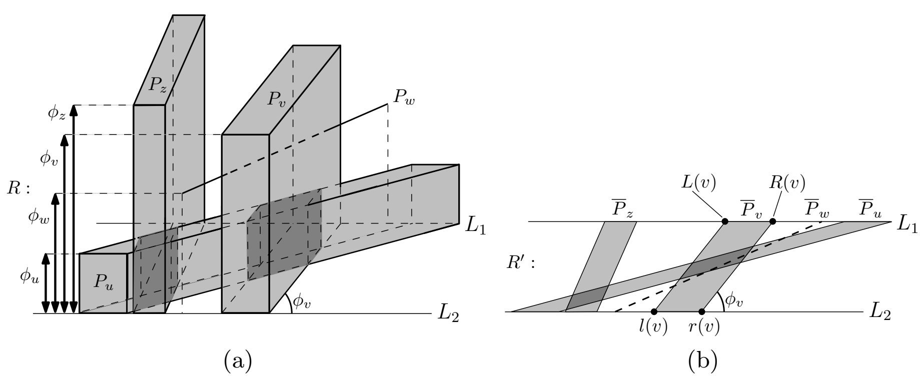

Fig. 1. (a) A parallelepiped representation R of the tolerance graph G and (b) the corresponding projection representation R' of G, where G is the induced path $P_4 = (z, u, v, w)$ with four vertices

two unbounded vertices u and v of G are never adjacent (even in the case where $\overline{P}_u$ intersects $\overline{P}_v$). In the following, we will call such a representation a *projection representation* of a tolerance graph. Note that $\overline{P}_u$ is a parallelogram (resp. a line segment) if u is bounded (resp. unbounded). The projection representation that corresponds to the parallelepiped representation of Figure 1(a) is presented in Figure 1(b). In the sequel, given a tolerance graph G, we will call a projection representation R of G a *canonical representation* of G, if R is the projection representation that is implied by a canonical parallelepiped representation of G. In the example of Figure 1, the projection representation R' is canonical, since the parallelepiped representation R is canonical as well.

Let R be a projection representation of a tolerance graph $G = (V, E)$. For every parallelogram $\overline{P}_u$ in R, where $u \in V$, we define by $l(u)$ and $r(u)$ (resp. $L(u)$ and $R(u)$) the lower (resp. upper) left and right endpoint of $\overline{P}_u$, respectively (cf. the parallelogram $\overline{P}_v$ in Figure 1(b)). Note that $l(u) = r(u)$ and $L(u) = R(u)$ for every unbounded vertex u. We assume throughout the paper w.l.o.g. that all endpoints and all slopes of the parallelograms in a projection representation are distinct [12, 13, 15]. For simplicity of the presentation, we will denote in the following $\overline{P}_u$ just by P_u in any projection representation.

Similarly to a trapezoid representation, we can define the relation $\ll_R$ also for a projection representation R. Namely, $P_u \ll_R P_v$ if and only if P_u lies completely to the left of P_v in R. Otherwise, if neither $P_u \ll_R P_v$ nor $P_v \ll_R P_u$, we will say that P_u *intersects* P_v in R, i.e. $P_u \cap P_v \neq \emptyset$ in R. Note that, for two vertices u and v of a tolerance graph $G = (V, E)$, P_u may intersect P_v in a projection representation R of G, although u is not adjacent to v in G, i.e. $uv \notin E$. Thus, a projection representation R of a tolerance graph G is *not* necessarily an intersection model for G.

In [11, 15] the *hovering set* of an unbounded vertex in a tolerance graph has been defined. According to these definitions, the hovering set depends on a particular representation of the tolerance graph. In the following, we extend this definition to the notion of *covering* vertices of an arbitrary graph G, which is independent of any representation of G.

Definition 3. *Let $G = (V, E)$ be an arbitrary graph and $u \in V$ be a vertex of G. Then,*

- *the set $\mathcal{C}(u) = \{v \in V \setminus N[u] \mid N(u) \subseteq N(v)\}$ is the* covering set *of u, and every vertex $v \in \mathcal{C}(u)$is a* covering vertex *of u,*
- *$V_0(u)$ is the set of connected components of $G \setminus N[u]$ that have at least one covering vertex $v \in \mathcal{C}(u)$ of u.*

In the following, for simplicity of the presentation, we may not distinguish between the connected components of $V_0(u)$ and the vertex set of these components. In the next definition we introduce the notion of the right (resp. left) border property of a vertex u in a projection representation R of a tolerance graph G. This notion is of particular importance for the sequel of the paper.

Definition 4. *Let $G = (V, E)$ be a tolerance graph, u be an arbitrary vertex of G, and R be a projection representation of G. Then, u has the* right *(resp.* left*)* border property *in R, if there exists no pair of vertices $w \in N(u)$ and $x \in V_0(u)$, such that $P_w \ll_R P_x$ (resp. $P_x \ll_R P_w$).*

We denote in the sequel by TOLERANCE the class of tolerance graphs, and we use the corresponding notations for the classes of bounded tolerance, cocomparability, and trapezoid graphs. Let $G \in$ TOLERANCE $\cap$ COCOMPARABILITY. Then G is also a trapezoid graph [7]. Thus, since TRAPEZOID $\subseteq$ COCOMPARABILITY, it follows that TOLERANCE $\cap$ COCOMPARABILITY = TOLERANCE $\cap$ TRAPEZOID. Furthermore, clearly BOUNDED TOLERANCE $\subseteq$ (TOLERANCE $\cap$ TRAPEZOID), since BOUNDED TOLERANCE $\subseteq$ TOLERANCE and BOUNDED TOLERANCE $\subseteq$ TRAPEZOID. In the following we consider a graph $G \in$ (TOLERANCE $\cap$ TRAPEZOID) $\setminus$ BOUNDED TOLERANCE, assuming that one exists, and our aim is to get to a contradiction; namely, to prove that (TOLERANCE $\cap$ TRAPEZOID) = BOUNDED TOLERANCE.

3 Main Results

In this section, we prove that for a graph $G \in$ (TOLERANCE $\cap$ TRAPEZOID) it follows that also $G \in$ BOUNDED TOLERANCE by making a slight assumption on the unbounded vertices of a projection representation R of G. In particular, we choose a certain unbounded vertex u in R and we "eliminate" u in R in the following sense: assuming that R has $k \geq 1$ unbounded vertices, we construct a projection representation R^* of G with $k-1$ unbounded vertices, where all bounded vertices remain bounded and u is transformed to a bounded vertex. In Section 3.1 we deal with the case where the unbounded vertex u has the right or the left border property in R, while in Section 3.2 we deal with the case where u has neither the left nor the right border property in R. Finally we combine these two results in Section 3.3, in order to eliminate all k unbounded vertices in R, regardless of whether or not they have the right or left border property.

3.1 The Case Where u Has the Right or the Left Border Property

In this section we consider an arbitrary unbounded vertex u of G in the projection representation R, and we assume that u has the right or the left border property in R. Then, as we prove in the next theorem, there exists another projection representation R^* of G, in which u has been replaced by a bounded vertex.

Theorem 1. *Let $G = (V, E) \in$ (*TOLERANCE $\cap$ TRAPEZOID*) $\setminus$ *BOUNDED TOLERANCE *with the smallest number of vertices. Let R be a projection representation of G with k unbounded vertices and u be an unbounded vertex in R. If u has the right or the left border property in R, then G has a projection representation R^* with $k-1$ unbounded vertices.*

3.2 The Case Where u Has Neither the Left Nor the Right Border Property

In this section we consider a graph $G \in$ (TOLERANCE $\cap$ TRAPEZOID) $\setminus$ BOUNDED TOLERANCE with the smallest number of vertices. Furthermore, let

R and R_T be a canonical projection and a trapezoid representation of G, respectively, and u be an unbounded vertex of G in R. We consider the case where G has no unbounded vertex in R with the right or the left border property (otherwise Theorem 1 can be applied). It can be proved that $V_0(u) \neq \emptyset$ and that $V_0(u)$ is connected (cf. [17]). Therefore, since u is not adjacent to any vertex of $V_0(u)$ by Definition 3, either all trapezoids of $V_0(u)$ lie to the left, or all to the right of T_u in R_T.

Consider first the case where all trapezoids of $V_0(u)$ lie to the *left* of T_u in R_T, i.e. $T_x \ll_{R_T} T_u$ for every $x \in V_0(u)$. It can be proved that $N(v) \neq N(u)$ for every unbounded vertex $v \neq u$ in R (cf. [17]). Denote by $Q_u = \{v \in V_U \mid N(v) \subset N(u)\}$ the set of unbounded vertices v of G in R, whose neighborhood set is included in the neighborhood set of u. Since no two unbounded vertices are adjacent, we can partition the set Q_u into the two subsets $Q_1(u) = \{v \in Q_u \mid T_v \ll_{R_T} T_u\}$ and $Q_2(u) = \{v \in Q_u \mid T_u \ll_{R_T} T_v\}$. Furthermore, it can be proved that $T_v \ll_{R_T} T_x \ll_{R_T} T_u$ for every $v \in Q_1(u)$ and every $x \in V_0(u)$ (cf. [17]). That is, $Q_1(u) = \{v \in Q_u \mid T_v \ll_{R_T} T_x \text{ for every } x \in V_0(u)\}$.

Consider now the case where all trapezoids of $V_0(u)$ lie to the *right* of T_u in R_T, i.e. $T_u \ll_{R_T} T_x$ for every $x \in V_0(u)$. Then, by performing vertical axis flipping of R_T, we partition similarly to the above the set Q_u into the sets $Q_1(u)$ and $Q_2(u)$. That is, in this (symmetric) case, the sets $Q_1(u)$ and $Q_2(u)$ will be $Q_1(u) = \{v \in Q_u \mid T_x \ll_{R_T} T_v \text{ for every } x \in V_0(u)\}$ and $Q_2(u) = \{v \in Q_u \mid T_v \ll_{R_T} T_u\}$.

We state now three conditions on G, regarding the unbounded vertices in R; the third one depends also on the representation R_T. Note that the second condition is weaker than the first one, while the third one is weaker than the other two. Then, we prove Theorem 2, assuming that the third condition holds. First, we introduce the notion of neighborhood maximality for unbounded vertices in a tolerance graph.

Definition 5. *Let G be a tolerance graph, R be a projection representation of G, and u be an unbounded vertex in R. Then, u is* unbounded-maximal *if there is no unbounded vertex v in R, such that $N(u) \subset N(v)$.*

Condition 1. *The projection representation R of G has exactly one unbounded vertex.*

Condition 2. *For every unbounded vertex u of G in R, $Q_u = \emptyset$; namely, all unbounded vertices are unbounded-maximal.*

Condition 3. *For every unbounded vertex u of G in R, $Q_2(u) = \emptyset$, i.e. $Q_u = Q_1(u)$.*

In the following of the section we assume that Condition 3 holds, which is weaker than Conditions 1 and 2. We present now the main theorem of this section. The proof of this theorem is based on the fact that G has simultaneously the two representations R and R_T.

Theorem 2. *Let* $G = (V, E) \in$ (Tolerance $\cap$ Trapezoid) $\setminus$ Bounded Tolerance *with the smallest number of vertices. Let* R_T *be a trapezoid representation of* G *and* R *be a projection representation of* G *with* k *unbounded vertices. Then, assuming that Condition 3 holds, there exists a projection representation* R^* *of* G *with* $k-1$ *unbounded vertices.*

3.3 The General Case

Recall now that Tolerance $\cap$ Cocomparability = Tolerance $\cap$ Trapezoid [7]. The next main theorem follows by recursive application of Theorem 2.

Theorem 3. *Let* $G = (V, E) \in$ (Tolerance $\cap$ Cocomparability)*,* R_T *be a trapezoid representation of* G*, and* R *be a projection representation of* G*. Then, assuming that one of the Conditions 1, 2, or 3 holds,* G *is a bounded tolerance graph.*

As an immediate implication of Theorem 3, we prove in the next corollary that Conjecture 1 is true in particular for every graph G that has no three independent vertices a, b, c such that $N(a) \subset N(b) \subset N(c)$, since Condition 2 is guaranteed to be true for every such graph G. Therefore, in particular, the conjecture is also true for the complements of triangle-free graphs.

Corollary 1. *Let* $G = (V, E) \in$ (Tolerance $\cap$ Cocomparability)*. Suppose that there do not exist three independent vertices* $a, b, c \in V$ *such that* $N(a) \subset N(b) \subset N(c)$*. Then,* G *is also a bounded tolerance graph.*

Definition 6. *Let* $G \in$ Tolerance $\setminus$ Bounded Tolerance*. If* $G \setminus \{u\}$ *is a bounded tolerance graph for every vertex of* G*, then* G *is a* minimally unbounded tolerance *graph.*

Assume now that Conjecture 1 is not true, and let G be a counterexample with the smallest number of vertices. Then, in particular, G is a minimally unbounded tolerance graph by Definition 6. Now, if our Conjecture 2 is true (see Section 1), then G has a projection representation R with exactly one unbounded vertex, i.e. R satisfies Condition 1. Thus, G is a bounded tolerance graph by Theorem 3, which is a contradiction, since G has been assumed to be a counterexample to Conjecture 1. Thus, we obtain the following theorem.

Theorem 4. *Conjecture 2 implies Conjecture 1.*

Therefore, in order to prove Conjecture 1, it suffices to prove Conjecture 2. In addition, there already exists evidence that this conjecture is true, as to the best of our knowledge all known examples of minimally unbounded tolerance graphs have a tolerance representation with exactly one unbounded vertex; for such examples, see e.g. [12].

4 Concluding Remarks and Open Problems

In this article we dealt with the over 25 years old conjecture of [10], which states that if a graph G is both tolerance and cocomparability, then it is also bounded tolerance. Specifically, we proved that the conjecture is true for every graph G, whose tolerance *representation* R of G satisfies a slight assumption, instead of making *any structural assumption* on G – as it was the case in all previously known results. Furthermore, we conjectured that any minimal graph G that is a tolerance but not a bounded tolerance graph, has a tolerance representation with exactly one unbounded vertex. Our results imply the non-trivial result that, in order to prove the conjecture of [10], it suffices to prove our conjecture. In addition, there already exists evidence in the literature that this conjecture is true [12].

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