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Discretization Schemes and Numerical Approximations of PDE Impainting Models and a comparative evaluation on novel real world MRI reconstruction applications

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Abstract *- While various PDE models are in discussion since ten years and are widely applied nowadays in image processing and computer vision tasks, including restoration, filtering, segmentation and object tracking the perspective adopted in the majority of the relevant reports is the view of applied mathematician, attempting to prove existence theorems and devise exact numerical methods for solving them. Unfortunately, such solutions are exact for the continuous PDEs but due to the discrete approximations involved in image processing the results yielded might be quite unsatisfactory. The major contribution of this paper is, therefore, to present, from an engineering perspective the application of PDE models in image processing analysing, from the algorithmic point of view, the discretization and numerical approximation schemes used for solving them. It is of course impossible to tackle all PDE models applied in image processing in this report from the computational point of view. It is, therefore, focused on image impainting PDE models, that is on PDEs, including anisotropic diffusion PDEs, higher order non-linear PDEs, variational PDEs and other constrained/regularized and unconstrained models, applied to image interpolation/ reconstruction. Apart from this novel computational critical overview and presentation of the PDE image impainting models numerical analysis, the second major contribution of this paper is to evaluate, especially the anisotropic diffusion PDEs, in novel real world image impainting applications related to MRI.*

Keywords *- Variational PDEs, Diffusion PDEs, PDE Discretization, Image Impainting, MRI Reconstruction.*

I. INTRODUCTION

While numerous papers dealing with image processing related PDEs from the applied mathematician's point of view already exist, proving for instance theorems for the associated continuous PDEs solutions existence or reviewing PDE models [1], there is lack of reports presenting the computational and numerical analysis perspective of the available PDE models. The major goal of this paper is to outline, from an engineering perspective, the discretization and numerical approximation algorithms for solving the most important PDE models in image impainting tasks [2]. This critical computational/ algorithmic overview is actually the first step towards numerical analysis of these PDE models.

Regarding image impainting [2], there is a variety of available PDE models proposed mainly to smooth and denoise images. In the last two decades, the use of nonlinear PDEs for image smoothing and denoising has met with tremendous success [1]. Before nonlinear PDEs were introduced, images were denoised by linear filtering, which is equivalent to using the noisy image as an initial condition for the heat equation. Although this method removes high frequency noise, it also badly blurs edges. To prevent blurring, a number of authors suggested using a nonlinear diffusion equation or a variational PDE model. Therefore, the main PDE categories investigated from the computational/algorithmic and engineering perspective in this paper are the diffusion and the variational PDE models.

Among the equations belonging in the first category the most famous example is the Perona-Malik equation [3,4]. On the other hand, among the most famous examples of PDEs belonging in the second category is the variational Mumford-Shah model [1,3,5] as well as the Total Variation (TV) model [6,7]. Although effective, the methods produce piecewise constant images, often giving "blocky" results. The Perona-Malik equation e.g behaves as a backwards heat equation and instantly creates jumps (i.e. shocks) in unpredictable locations [8]. Such results occur either because of the PDE model which might be not representative of the image dynamical system or because of the discretization and numerical approximation schemes involved. The mainstream research on PDEs in image processing is focused mainly on improving the PDE models rather than on investigating the discretization / numerical approximation algorithms

In an attempt to improve upon the piecewise constant images resulting from second order image diffusions, a number of fourth order diffusions have been suggested for image denoising mainly. Examples include the 'Low Curvature Image Simplifier' (LCIS) equation of Tumblin and Turk [8,9] as well as similar higher order PDE models [10]. Other mainstream such attempts include the imposing of constraints in the diffusion PDE models [11].

Regarding the attempts for improvements in the second PDE main category, namely variational PDEs, herein numerically examined, the regularization approach [7] will be computationally reviewed in this paper.

The above mentioned PDE models are the most important mainstream PDE models for image impainting, and the contribution of this paper lies on the, from the engineering perspective, numerical analysis of all these models.

The focus of the herein presentation is, however, the discretization and numerical approximation algorithms involved in these PDE models. Although the mainstream discretization algorithm is, as mentioned above the finite difference method, there are many variations of it [8] that will be herein reviewed as well as the more recent Radial Basis Functions (RBF) discretization schemes that will be analysed in the context of the above PDE models, along with other important discretization and numerical approximation models, including [12]-[15].

Apart from this numerical analysis overview of the most important PDE models in image impainting, the second major goal of the paper is to comparatively evaluate mainly anisotropic diffusion PDE models in innovative MRI reconstruction applications. Although, the performance of the second order diffusion PDE models mainly in image denoising is known to a certain extend, other recent models, like the fourth order diffusion PDEs have almost unknown performance. The goal of the comparative study herein conducted is, therefore, first to evaluate newer more promising diffusion PDE models in novel MRI reconstruction problems. The real world complex task selected as a platform for evaluating the previously mentioned PDE models and the associated discretization schemes is the MRI reconstruction from sparsely sampled k-space. The goal in such a case is to reduce the measurement time by omitting as many scanning trajectories as possible. This approach, however, entails underdetermined equations and leads to poor image reconstruction due to Nyquist sampling theorem possible violations.

The organization of this paper is as follows. In section II an overview of the principles and the concepts of the PDE models applied to image impainting tasks is presented. In section III special attention is paid to the discretization schemes and numerical approximations of such PDE models. Section IV illustrates a comparison between second and fourth order PDE models in the novel application of MRI reconstruction from sparsely sampled k-spaces. Finally, section V concludes the paper.

II. OVERVIEW OF IMAGE IMPAINTING PDE MODELS FROM AN ENGINEERING POINT OF VIEW

Partial Differential Equations have led to an entire new field in image processing and computer vision. They offer several advantages:

- Better and intuitive mathematical modeling, connection with physics and better approximation to the Euclidean geometry of the problem. Deep mathematical results with respect to well-posedness are available, such that stable algorithms can be found.
- They allow a reinterpretation of several classical methods under a novel unifying framework. This includes many well-known techniques such as Gaussian convolution, median filtering and morphological operations of dilation/erosion.
- This understanding has also led to the discovery of new methods. They can offer more invariances than classical techniques, or describe novel ways of shape simplification, structure preserving filtering, and enhancement of coherent line-like structures.
- The PDE formulation is genuinely continuous. Thus, their approximations aim to be independent of the unifying grid and may reveal good rotational invariance.

Moreover, the majority and the most useful image analysis techniques are nonlinear, which is due to the inability of linear systems to successfully model important problems. The most known vision problem modeled via PDEs is that of multiscale analysis, which is a useful and often required framework for many tasks such as feature/ object detection, motion detection, stereo and multi- band frequency analysis.

Consider a multiscale operator T_t mapping an input image f to an output image $T_t(f)$, which results from the interaction of f with some kernel function dependent on a continuous scale parameter $t \geq 0$, i.e. $T_t(f)(x,y) = u(x,y,t)$. The scale-space function $u(x,y,t)$ holds all the history of transforming f though all the scales and van be viewed as the output of T_t at any fixed scale t. The evolution of u in scale-space as a continuous dynamical system can be modeled by evolution PDEs of the type $u_t(x,y,t) = function(u_{xx}, u_{xy}, u_{yy}, u_x, u_y, u, x, y, t)$ and u can be viewed as the solution of the PDF with initial condition $u(x,y,0) = f(x,y)$.

The most characteristic PDEs for Image Impainting are:

A. Linear heat-diffusion PDE.

The most investigated PDE method for smoothing images is to apply a linear diffusion process for modeling Gaussian scale-space [16]. The convolution of an image with a Gaussian function of increasing variance is equivalent from a physical point of view with linear diffusion filtering. The connection of Gaussian convolution and linear diffusion filtering extends its limits to **multiscale analysis**. When there is not clear in advance which is the right scale, it is desirable to have an image representation at multiple scales. Diffusion could be thought as a physical process that equilibrates concentration differences without creating or destroying mass. The equilibration property is expressed by *Fick's* law:

$$j = -D \cdot \nabla u \tag{1}$$

which states that a concentration gradient ∇u causes a flux j, aiming to compensate the gradient. The relation between ∇u and j is described by the *diffusion tensor* D, a positive definite symmetric matrix. The case where ∇u and j are parallel is called isotropic. Then the diffusion tensor may be replaced by a positive scalar-valued *diffusivity* g. In the general anisotropic case, ∇u and j are not parallel.

The observation that diffusion does only transport mass without destroying it or creating new mass is expressed by the *continuity equation*:

$$\partial_t u = -div(j) \qquad (2)$$

where t denotes the time. By connecting Fick's law with the continuity equation we end up with the *diffusion equation*:

$$\partial_t u = div(D \cdot \nabla u) \qquad (3)$$

Equation (3) appears in many physical transport processes. In image processing we may identify the concentration with the gray value at a certain location.

B. Anisotropic Diffusion PDE Models

Since linear filtering causes edge blurring and linear shifting, the development of anisotropic nonlinear diffusion PDEs for multiscale directional image smoothing and edge detection was motivated. Perona and Malik ([4]) proposed a nonlinear diffusion method for avoiding the blurring and localization problems of linear diffusion filtering (hence T_t nonlinear). This Perona-Malik scheme appears to be the finite differences discretization of a nonlinear PDE not followed by a theory of well-posedness. It was known that, despite its success at its intended purpose the scheme is very sensitive to the presence of noise and the choice of parameters such as the resolution of the digital image - a fact intimately connected with the lack of a continuum PDE theory. The work of Lions et al. [17] replaced the Perona-Malik scheme with one that has all the desirable characteristics of the original, as well as a rigorously established continuum limit. A scale – space is an image representation at a continuum of scales, embedding the image into a family of gradually simplified versions of it [17]. The practical implication is much more stable behavior with respect to the presence of noise and different resolutions.

Computationally, solving the modified Perona-Malik anisotropic diffusion equation, mainly following Rothe's approximation in time and finite element method in space, involves the PDE

u_t - div (g(|grad G_sigma * u|) grad u) = f (u_0 - u)

together with zero Neumann boundary conditions and initial condition representing the processed image. Here, g(s) tends to 0 for s tends to infinity. It causes the selective smoothing of the image regions and keeping of the edges on which the 'gaussian gradient' is large (G_sigma is smoothing kernel of the convolution). Such image analysis is included in the so-called nonlinear scale space theory.

C. Differential Morphology PDE Models

Independently from the Gaussian scale-space ideas and their anisotropic/nonlinear versions, nonlinear PDEs were developed to model multiscale morphological dilations/ erosions or distance propagation in images [18]-[21]. These morphological PDEs are related to a broader classes of nonlinear PDEs that can model nonlinear dynamics in image analysis. The PDEs governing multiscale morphology and curve evolution are of Hamilton-Jacobi type and are related to the eikonal equation of optics and are solved via weighted distance transforms, which are bandpass slope filters. These slope filters may be implemented sequentially and in parallel.

Two-dimensional max/min-sum nonlinear difference equations model the space dynamics of morphological systems and are related to numerical solutions. Moreover, nonlinear signal transforms, called slope transforms, can analyze morphological systems in a transform domain, in a way conceptually similar to the application of Fourier transform to linear systems.

D. Variational PDE Models

The variational approach to the image denoising problem seeks to exhibit the "restored" image as the minimizer of a functional defined over the space of all images. The first task is clearly to decide which space of functions to take images from. For example, Sobolev spaces are ill suited for this purpose since their elements cannot have discontinuities. Such discontinuities need to be allowed because one of the most important features of images, namely "edges" correspond squarely to this type of behavior.

A variational approach has been proposed [5] for the solution of the image segmentation problem, where the segmentation is obtained by finding the minimizer of an energy, given an original image. The correct space of functions for minimizing the energy, turns out to be a subset of functions of bounded variation. The Mumford-Shah model is a non-typical variational problem, whose analysis led to a wealth of new mathematics. Numerical implementation of the Mumford-Shah model has also been a subject of intense mathematical research. The energy is very difficult to handle since it requires minimization over subsets. The work of L. Ambrosio and V. M. Tortorelli [1] has rigorously shown how to approximate it in the sense of Gamma convergence by elliptic functionals. In a different vein, the work of T. Chan and L. Vese [1] has shown how the level set method of S. Osher and J. Sethian can be effectively utilized in the minimization of these types of energies.

Another successful example of the variational and PDE method is the Total Variation (TV) minimization [6]. An improved version of the latter technique that is based on the Connectivity Principle is the Curvature Driven Diffusions (CCD) inpainting scheme [3,22].

III. METHODS FOR PDEs NUMERICAL SOLUTION AND DISCRETIZATION

The numerical solution of partial differential equations (PDEs) has been dominated by either finite difference methods (FDM), finite element methods (FEM) and finite volume methods (FVM). These methods can be derived from the assumptions of the local interpolation schemes. These methods require a mesh to support the localized approximations; the construction of a mesh in three or more dimensions is a non-trivial problem. Typically with these methods only the function is continuous across meshes, but not its partial derivatives.

In practice, only low order approximations are used because of the notorious polynomial snaking problem. While higher order schemes are necessary for more accurate approximations of the spatial derivatives, they are not sufficient without monotonicity constraints. Because of the low order schemes typically employed, the spatial truncation errors can only be controlled by using progressively smaller meshes. The mesh spacing, h, must be sufficiently fine to capture the functions of the partial derivative behavior and to avoid unnecessarily large amounts of numerical artifacts contaminating the solution. Spectral methods while offering very high order spatial schemes typically depend upon tensor product grids in higher dimensions [23].

The last decade the idea of using *meshless* methods for the numerical solution of PDEs has received much attention [24]-[25] and methods based on wavelets [26]-[27]. The underlying idea with *meshless* methods is the wish to design a numerical algorithm for PDEs without requiring a mesh as in FEM and similar methods.

A. Solving PDEs with Radial Basis Functions (RBF)

The idea of solving numerically PDEs based on Radial Basis Functions (RBFs) mostly have dealt with elliptic problems, although some efforts have been made to solve time dependent parabolic or hyperbolic problems

When RBFs were first introduced to scattered data fitting and to numerical solution of PDEs [23]. This was done in the form globally supported RBFs and specifically of *multiquadrics (MQ)* $\phi(r)=\sqrt{r^2+c^2}$ of thin plate splines $\phi(r)=r^2\log r$, or Gaussians $\phi(r)=e^{-c^2r^2}$, where $r=\|x\|$ with $c\neq 0$ is a parameter.

For the solution of the scattered data fitting problem [28] an RBF based expansion of the form

$$s(x) \ = \ \sum_{j=1}^{n} c_j\phi\left(\left\|x-x_j\right\|_2\right),$$

and then, the coefficients c_j are determined by satisfying the interpolation conditions

$$s(x_i) \ = \ f(x_i), \quad i=1,2,\ldots,n$$

where f is a known function that generates the data to be fitted. There exists a trade-off principle, which says that the spectral convergence is achieved at the cost of instability.

Moreover, gradient based methods are ill-conditioned, and converge rapidly only under certain restricted conditions. In addition, gradient methods pose the risk of being trapped in a local minimum, rather than in the global minimum.

Galperin and Zheng [29] argue that all collocation methods are intrinsically ill conditioned. Ill-posed and badly formulated problems can possess-equivalent solutions that represent physical reality despite the mathematical nonexistence of an exact solution. Only Galperin, Pan, and Zheng [30] have used global optimization on a few limited problems, with extra-ordinary results.

Although it is clear that the numerical solutions of PDE, ODE, integral, and integro-differential equations would greatly benefit from the global optimization, the major implementation impediment is the lack of robust multi-parameter global optimization software. Unfortunately, gradient based methods are ill-conditioned, and converge rapidly only under certain restricted conditions. In addition, gradient methods pose the risk of being trapped in a local minimum, rather than in the global minimum. Ferrari and Galperin [31] have published a software package of a fast one-dimensional adaptive cubic algorithm. It is hopeful that fast multi-dimensional global optimization software packages would be developed soon.

Therefore there are serious limitations to the applicability of global methods and for large dimension problems the solution should employ localization schemes ([32]). The localization of the basis functions leads to locally (compactly) supported RBFs. One of the most popular compactly supported RBFs has been proposed by Wendland ([32]) in order to use compactly supported RBFs in the context of scattered data fitting or to solve PDEs, an hierarchical strategy has been developed leading to a multilevel algorithm, in which residuals are fitted iteratively and are used to update the solution. Then, he computational complexity is held proportional to the number of points used at the computational mesh, while achieving at least linear convergence. The addition of an approximate smoothing is also needed that ensures superlinear convergence in an entire scale of Sobolev (or more generally Besov) spaces ([32]).

B. Solving PDEs with Wavelets

The wavelet approach and the associated multiresolution analysis has the following advantages ([27]):

- provide nice approximation spaces V_j, W_j suitable for the computation of an approximate solution to problems in which small-scale structures are localized in space and whose location vary in time
- leads to fast hierarchical algorithms with $O(N\log N)$ order of complexity, since multiresolution is the keystone for the design of fast algorithms and
- characterized by nice properties that ensure efficient inversion.

The strength of the wavelet-based approaches appears to be clear for the problems with many spatial scales, distributed unevenly over the domain.

C. Solving PDEs with Multi Dimensional Wave Digital Filters (MD-WDFs)

An alternative approach for simulating and solving PDEs in discrete space-time has been proposed by Fettweis and is based on properties of Kirchhoff networks [33]-[37]. This technique involves firstly finding a multi-dimensional lumped electrical network which represents the behavior of the linear or non-linear system. From this network, a discrete-time equivalent is developed in a set of computational nodes represented by a multi-dimensional wave digital (MDWD) filter. Those computational nodes can not be mapped one to -one to processor elements. A Locally Parallel-Globally Sequential approach may be used for the partitioning of the set of nodes in blocks and the final efficient implementation of the MDWD model [38].

From a computational point of view, this method allows a very great level of parallelism because each point in the n-D grid can be updated simultaneously if sufficient computing resources are available. Generally, this approach can be applied to problems with finite propagation speed, i.e. systems described by sets of hyperbolic partial differential equations. This represents a large range of problems. Elliptic and Parabolic PDEs can be dealt with MD-WDFs too, after some modifications to the equations. This technique has been successfully applied to the solution of wave problems, like Maxwell's equations, fluid, acoustical, and transmission-line problems, but not to image processing yet.

The main distinct advantages of an MDWD model are:

- High accuracy due to the use of the WDF structure, which is known to have low roundoff noise characteristics [33]
- Internal multi-dimensional passivity (reflecting the passivity of the system to be modeled) which guarantees numerical robustness,
- Guaranteed numerical stability properties required
- Generation of a 2nd order difference equation relating the value of a point on the grid to previous values of nearest-neighbor grid points (locality property).
- The use of massive parallelism.

while its limitations are:

- In general it induces a non standard (non-rectangular) sampling grid into the simulation space
- It is not directly applicable to parabolic and elliptic PDEs

IV. EXPERIMENTAL STUDY

An experimental study has been conducted in order to evaluate the diffusion PDE (the 2nd order model of Perona-Malik [4] and the 4th order model in [10]), out of the models outlined in sections II and III above, when applied to the novel application of MRI reconstruction from sparsely scanned k-spaces. A thorough experimental study of the discretization schemes and PDE numerical approximation techniques is under the way by the authors regarding the same reconstruction task as well as different real world MRI reconstruction problems, including reconstruction of compressed MRI and reconstruction of MRI images when communicated through AWGN channels with fading effects. In the herein limited study, however, the second as well as the fourth order diffusion PDEs are evaluated. The methods involved have been applied to an MRI image database which has been downloaded from the Internet, namely, the Whole Brain Atlas http://www.med.harvard.edu/AANLIB/home.html (copyright © 1995-1999 Keith A. Johnson and J. Alex Becker). These images have 256 by 256 dimensions. The k-space data for these images have been produced by applying the 2D FFT to them. Radial and spiral trajectories have been used to scan the resulted 256 X 256 complex array of k-space data. In the case of radial scanning 4 X 256 = 1024 radial trajectories are needed to completely cover k-space. On the other hand, in the case of spiral scanning 60 spirals are enough for attaining a good image reconstruction. In order to apply the PDE techniques involved in this study, the k-space has been sparsely sampled using 128 only radial trajectories in the former case and 30 only spiral trajectories in the latter.

Moreover, the simplest "interpolation" approach, namely filling in the missing samples in k-space with zeroes and then reconstructing the image, has been invoked. These methods (2nd and 4th order PDEs as well as the zero-filling based reconstruction) have been implemented in the MATLAB programming language and all simulations have been carried out using the MATLAB programming platform.

Concerning the measures involved to quantitatively compare the performance of the various interpolation techniques, we have employed the usually used Sum of Squared Errors (SSE) between the original MRI image pixel intensities and the corresponding pixel intensities of the reconstructed image as well as the RMS error in dB.

The quantitative results obtained by the different interpolation methods involved are outlined in table 1.These results show a superiority of the fourth order PDE in terms of MRI image reconstruction performance.

Test	**Sampled**	**zero-filled reconstruction**		**2nd order diffusion PDE [4]**		**4th order PDE model in [10]**	
Picture	**Trajectories**	**SSE**	**dB**	**SSE**	**dB**	**SSE**	**dB**
tc1	**128 rad.**	**3.71 E3**	**15.26**	**3.28E3**	**16.57**	**2.32E3**	**18.44**
tc1	**30 spir.**	**1.61 E4**	**3.76**	**9.87E3**	**8.02**	**8.35E3**	**9.78**
tl4	**128 rad.**	**2.49 E3**	**15.51**	**1.60E3**	**18.82**	**0.9E3**	**19.23**
tl4	**30 spir.**	**1.03 E4**	**4.11**	**5.34E3**	**8.9**	**4.21E3**	**10.56**
dg1	**128 rad.**	**3.38 E3**	**10.04**	**2.42E3**	**12.45**	**1.64E3**	**12.77**
038	**128 rad.**	**2.47 E3**	**14.32**	**1.39E3**	**18.13**	**0.95E3**	**18.32**

Table 1. The quantitative results with regards to reconstruction performance of the various methodologies involved

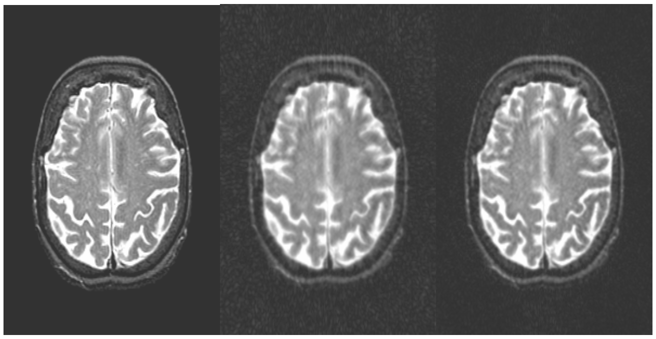

Fig.1. The 4th order PDE model reconstructed image showing a normal brain slice 38- (http://www.med.harvard.edu/AANLIB/cases/caseM/mr1/038.html), the zerofilled image reconstruction and the 2nd order diffusion PDE reconstructed image

V. CONCLUSIONS AND FUTURE TRENDS

A critical overview of the PDE models applied to image impainting problems and their most prominent discretization schemes and numerical approximations is herein presented from an engineering perspective. Additionally, a limited experimental study concerning the novel application of PDE models in MRI reconstruction from sparsely sampled scans is herein considered. A detailed experimental study on this as well as on other real world innovative applications of PDE models in MRI reconstruction is under the way by the authors. Moreover, the application of the MD-WDF approach in numerically approximating PDEs in MRI reconstruction is the major future goal of the authors.

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