On inefficient temporal graphs

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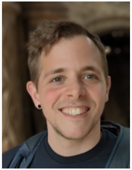
Algorithmic Aspects of Temporal Graphs VI, Paderborn, July 10, 2023

[1] Supported by the RIN Tremplin Région Normandie project DynNet.
[2] Supported by the Leverhulme Trust International Professorship in Neuroeconomics.

Context

Temporal graphs

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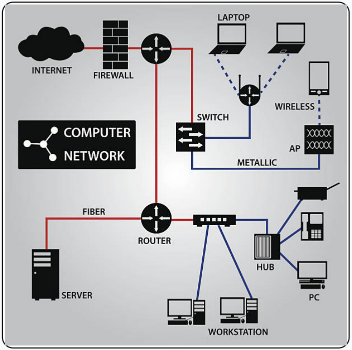
Temporal graphs

- faulty networks



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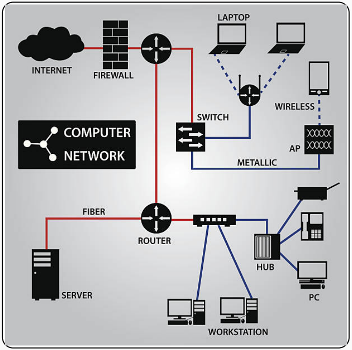
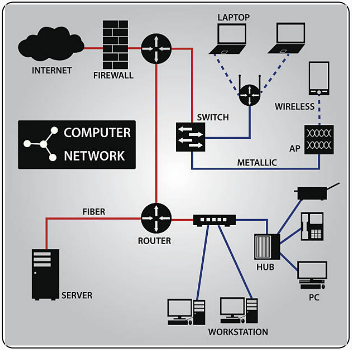


Temporal graphs

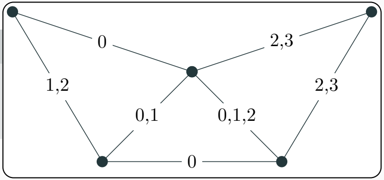
- faulty networks
- swarms of drones
- interacting population



Preliminaries

Definition temporal graph

Edges appear and disappear
over the lifetime of the graph



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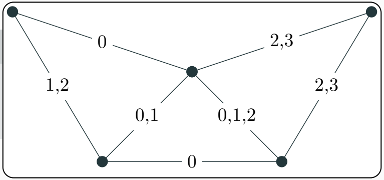

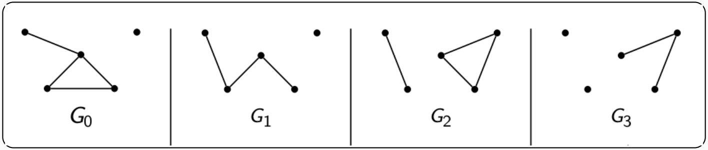

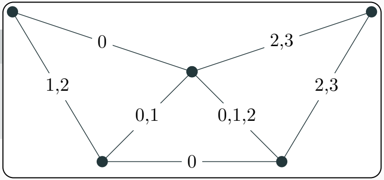

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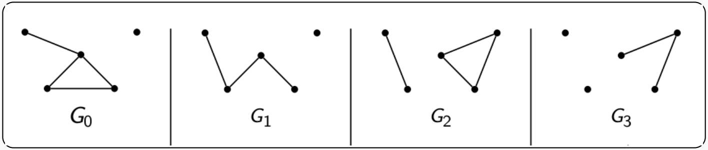

Definition footprint (or underlying graph)

Union of all snapshots G_i

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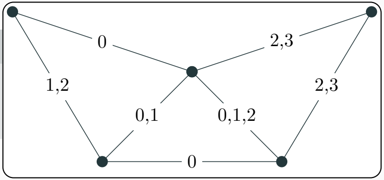

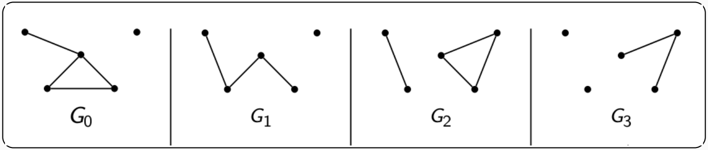

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Path in temporal graph with increasing labels on the edges

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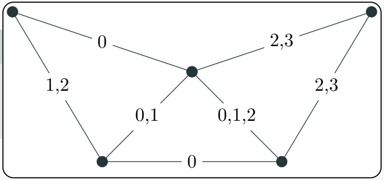

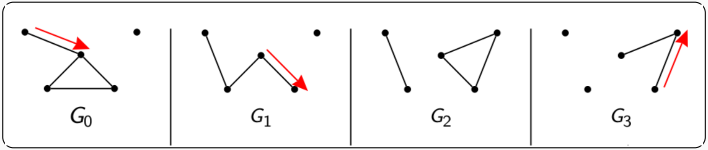

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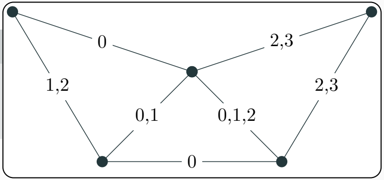

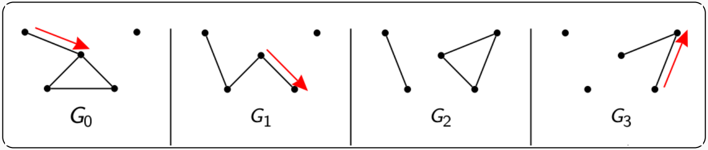

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Path in temporal graph with increasing labels on the edges

Definition temporal connectivity ($\mathcal{TC}$)

A temporal graph is temporally connected
iff for all pairs of vertices (u, v) there exists a journey $u \rightsquigarrow v$

Preliminaries

Temporal graph design problem

Input: graph $G = (V, E)$, temporal graph property $\mathcal{P}$

Question: does a labelling $\lambda : E \to \mathbb{N}$ exist such that temporal graph $\mathcal{G} = (G, L)$ have property $\mathcal{P}$?

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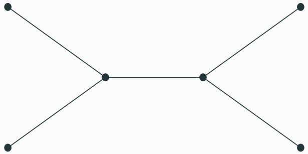
- $\mathcal{TC}$ with amount of labels $\leq 2n - 2$?



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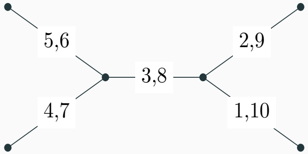

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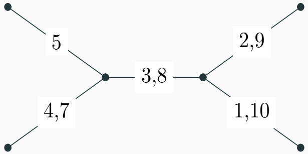

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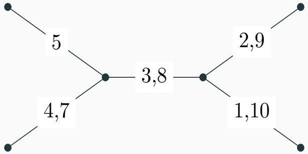

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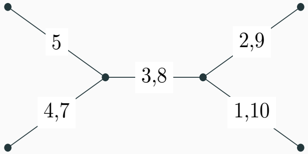

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Remark

The amount of labels used in these problems is always minimized or given

Sparse labelling problem

Input: graph G, integer k

Question: Does a labelling λ using at most k labels exist such that temporal graph $\mathcal{G} = (G, \lambda)$ is $\mathcal{TC}$?

Problem formulation

Dense labelling problem

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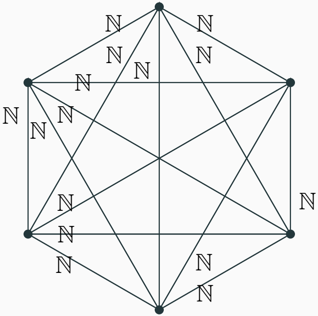

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- A label is **necessary** iff reachability reduces when removed
- A labelling is **necessary** iff it consists of only necessary labels
- A labelling is **proper** iff incident edges have distinct labels

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Motivation

- Represents potential power of an adversary trying to waste precious network resources (even if restricted)
- Worst case scenarios for temporal spanners and greedy algorithms

Measures

of a temporal graph $\mathcal{G}$ [6]:

- **Temporal cost** T: total amount of labels in $\mathcal{G}$

Global and local measures

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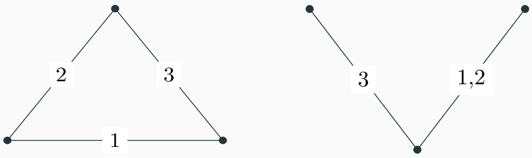

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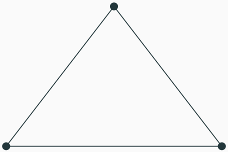

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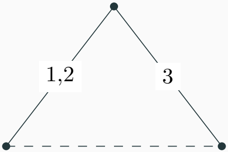

Bounds

Bounds on largest τ^+ and T^+

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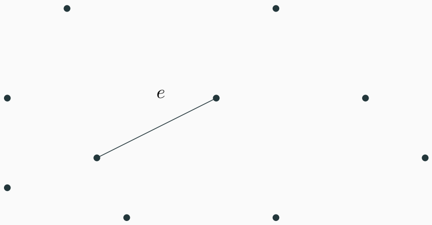

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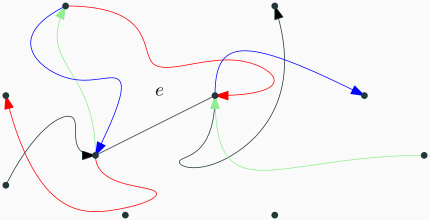

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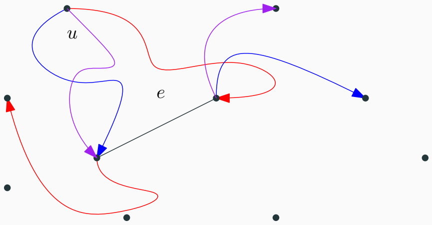

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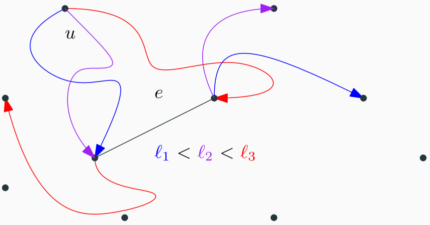

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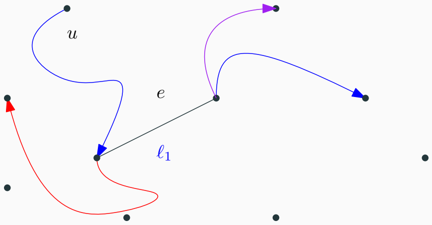

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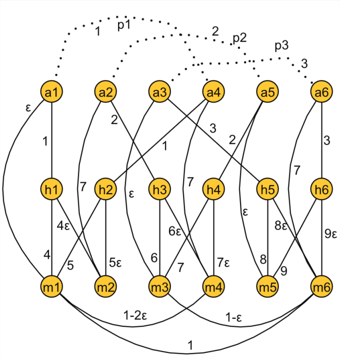
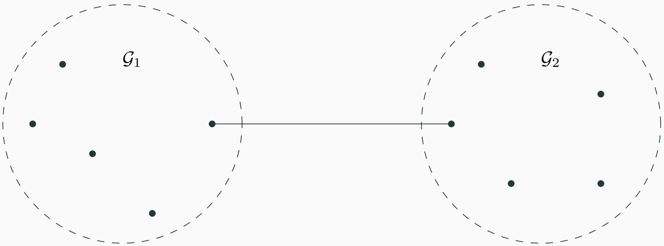
- 2 ? $\leq \tau^+$ and $\tau^+ \leq n-1$
- $\frac{1}{18}n^2 + O(n) \leq T^+$ [1] and $T^+ < (n-1)\binom{n}{2}$



The simple case of trees

Definition

A bridge edge is an edge which disconnects the graph when removed



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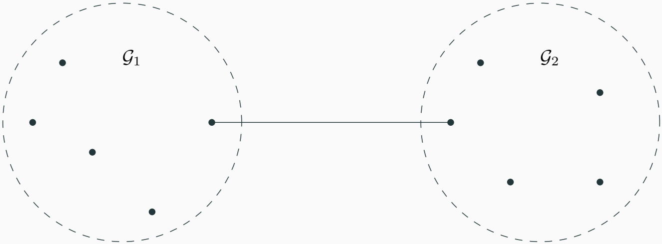
Lemma

In any necessary labelling, there are at most 2 labels on a bridge edge



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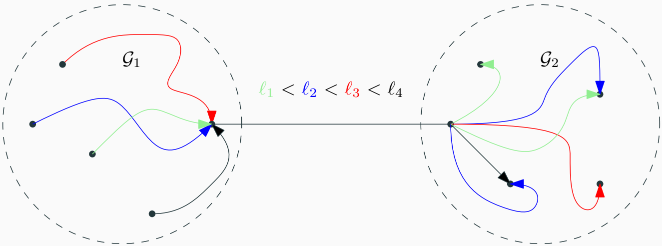

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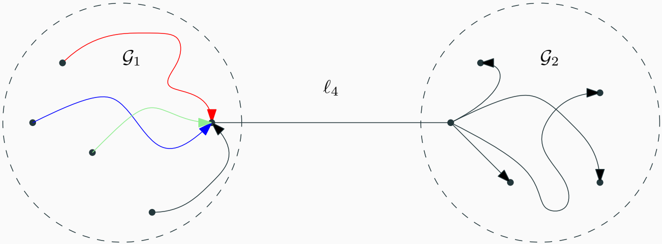

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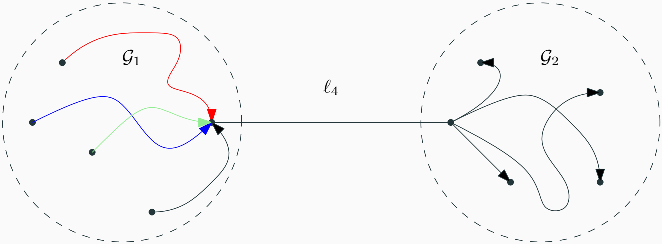

Theorem

For any tree graph on n vertices, $\tau^+ = 2$ and $T^+ = 2n - 3$

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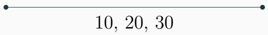
Ad-hoc construction for $\tau^+ \geq 3$

- suppose some edge e exists with 3 necessary labels;



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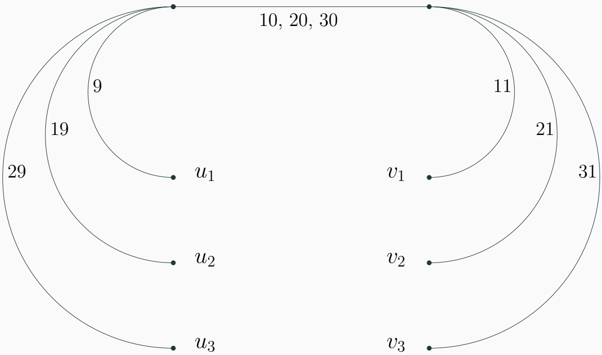

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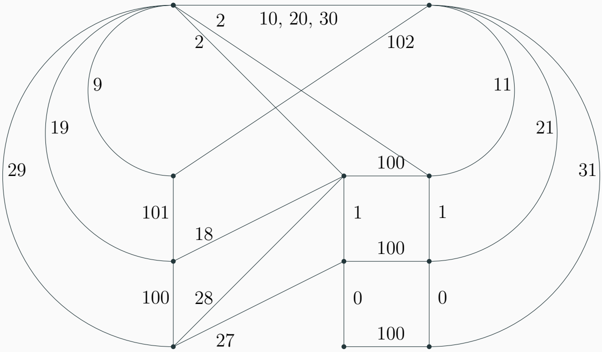
- add edges/labels to make sure graph is $\mathcal{TC}$



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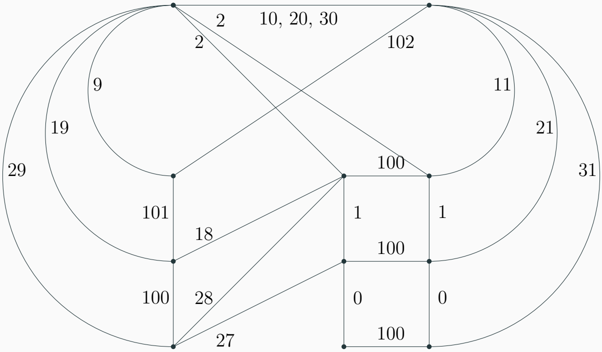
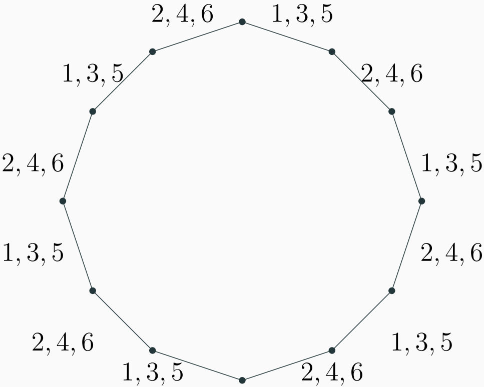


Theorem

The ad-hoc construction allows for $\tau^+ \geq \frac{1}{3}n$ and $T^+ \geq \frac{1}{18}n^2 + O(n)$

Odd/even alternating labelling



Theorem

The odd/even alternating labelling allows for $\tau^+ = \frac{1}{4}n$ and $T^+ = \frac{1}{4}n^2$

Remark

Contrary to trees, cycles allow for large τ^+ and T^+, but can we do better?

Necessary labelling generator

Idea

Extend STGen [3], a happy labelling generator using techniques based on matchings, isomorphisms, and automorphisms, to generate, given a graph, all its proper necessary labellings

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Some details

- Non-necessary labellings and $\mathcal{TC}$ labellings are "final", efficiently cutting branches of search space;
- Amortized constant time complexity for $\mathcal{TC}$ testing and necessity testing;
- Freely available at https://gitlab.com/echrstnn/max-temporality (Rust);

Necessary labelling generator

Idea

Extend STGen [3], a happy labelling generator using techniques based on matchings, isomorphisms, and automorphisms, to generate, given a graph, all its proper necessary labellings

Some details

- Non-necessary labellings and $\mathcal{TC}$ labellings are "final", efficiently cutting branches of search space;
- Amortized constant time complexity for $\mathcal{TC}$ testing and necessity testing;
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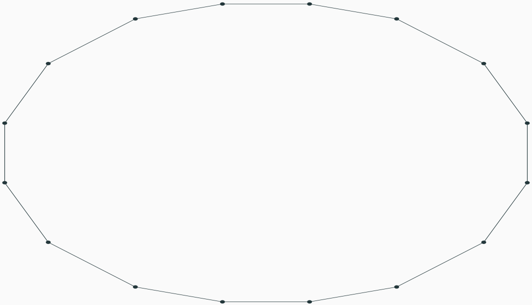
Result

Enumerated all necessary labellings of cycles of size up to $n = 14$ included, giving us the intuition for the following (empirically optimal) labelling

Odd/even distributing labelling

Definition Odd/even distributing labelling

- Let $L_1 = (1, 3, 5, ..., n-1)$, $L_2 = (2, 4, ..., n-2)$ and $P_1, P_2 = \emptyset$;
- Choose some edge e, and let $e_\ell, e_r = e$;
- Do $\frac{n}{2}$ times:
  - Assign P_1 to both edges e_ℓ and e_r;
  - Distribute L_1 to e_ℓ and e_r, move $\min(L_1)$ to P_1 and remove $\max(L_1)$;
  - Set e_ℓ and e_r to the next clockwise and counter-clockwise edges;
  - Swap L_1 and L_2, as well as P_1 and P_2;
- Assign P_1 to e_ℓ, as well as label $\max(P_1) + 2$



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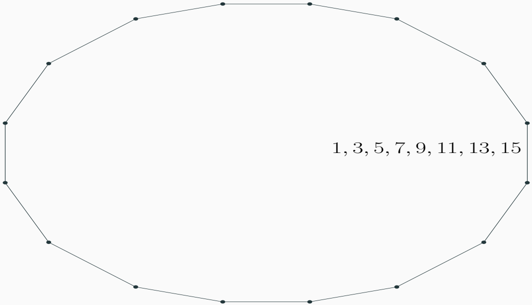

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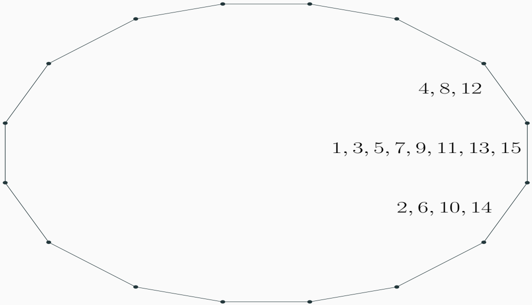

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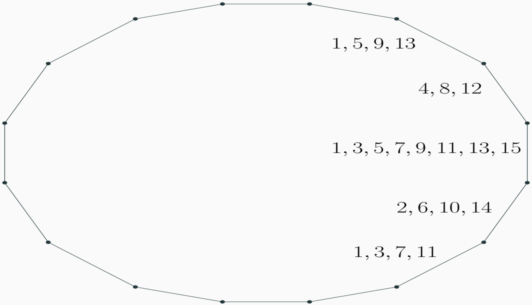

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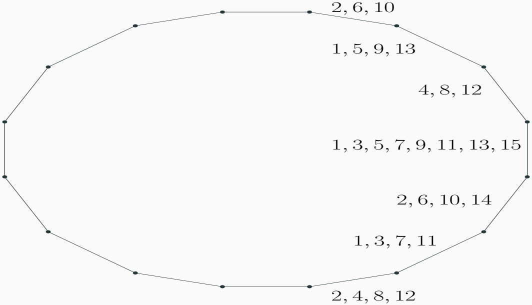

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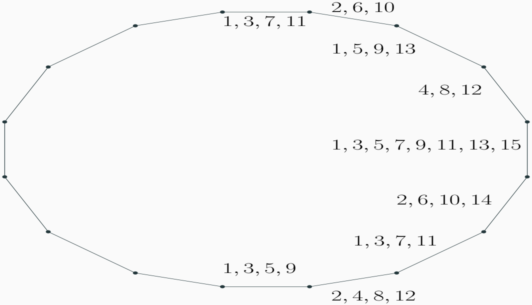

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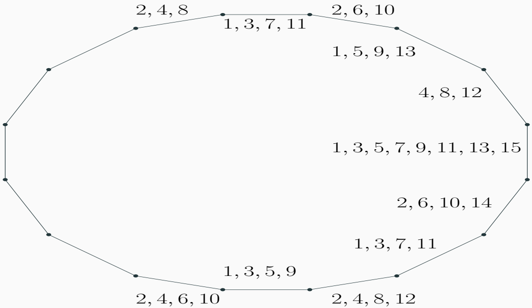

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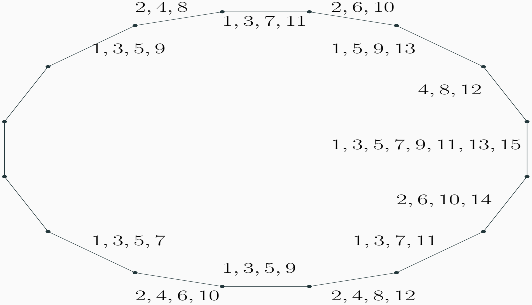

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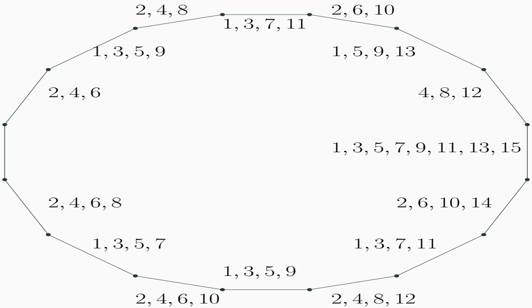

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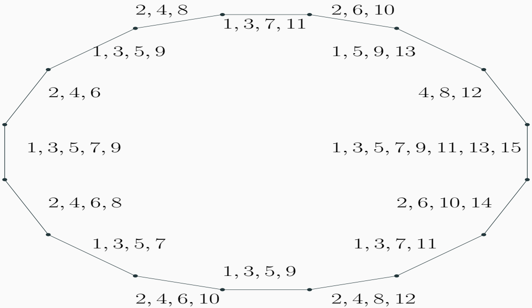

Correctness proof of odd/even distributing labelling

Theorem

The odd/even distributing labelling induces $\tau^+ \geq \frac{1}{2}n$ and $T^+ \geq \frac{1}{4}n^2 + \mathbf{1}$

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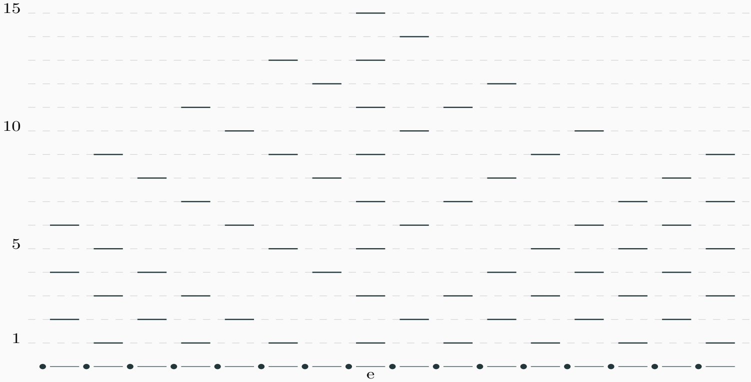
Link Stream (Latapy *et al.* [5])

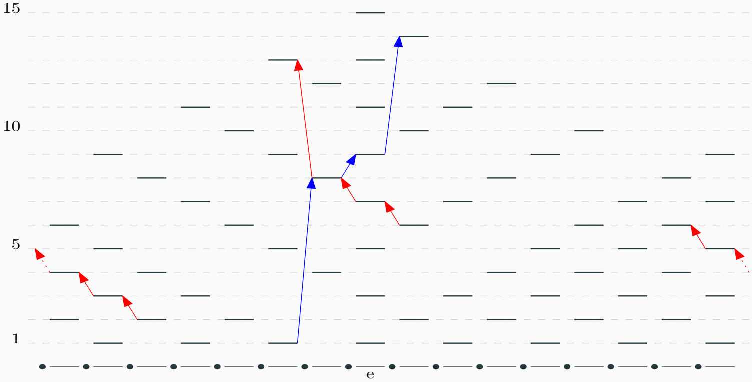
Use one axis for the vertices, and a second axis for time, allowing one to plot the time-edges (and journeys) in this two-dimensional space



Definition

A journey is:

- (counter-)clockwise iff it only uses edges towards the right (*resp.* left);



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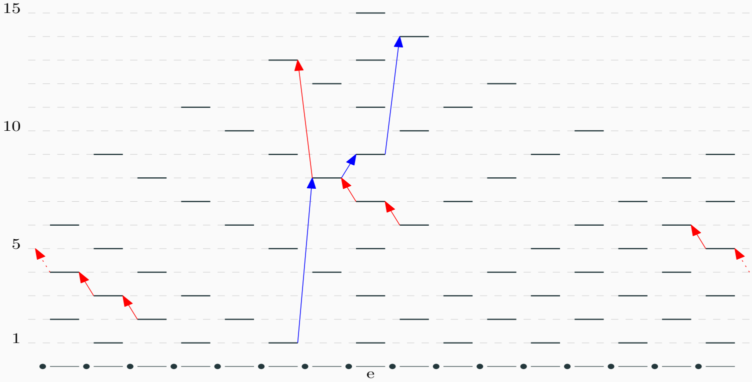

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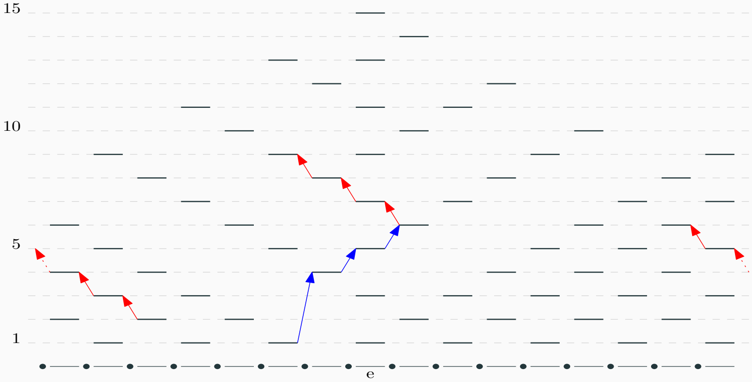

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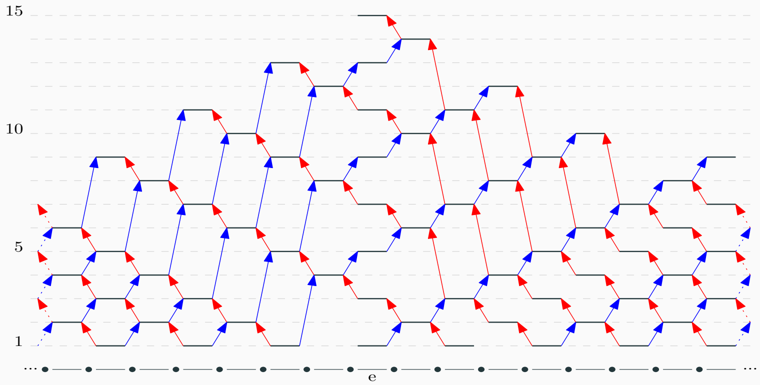
A journey is:

- (counter-)clockwise iff it only uses edges towards the right (*resp.* left);
- prefix-foremost iff it always uses the earliest edges possible;
- dominant iff no (same direction) journey covers its vertices or more

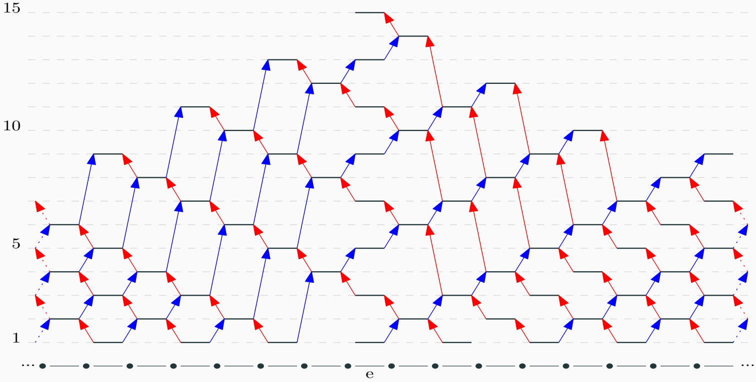


Correctness proof of odd/even distributing labelling

Lemma

A pair of clockwise and counter-clockwise journeys is necessary if:

- both start at some same vertex v;
- both are prefix-foremost;
- both are a suffix of a dominant journey;
- together they cover the whole vertex set without crossing



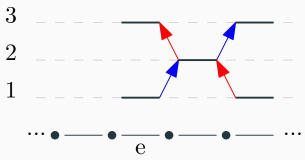
Correctness proof of odd/even distributing labelling

Theorem

The odd/even distributing labelling is necessary for all even cycles

Proof idea by induction

- base case: prove it is a necessary labelling for C_4 with Lemma
- inductive step (C_n to C_{n+2}): adds a “layer” on the link stream, lengthening journeys by 1, and adding journeys obeying Lemma $\square$



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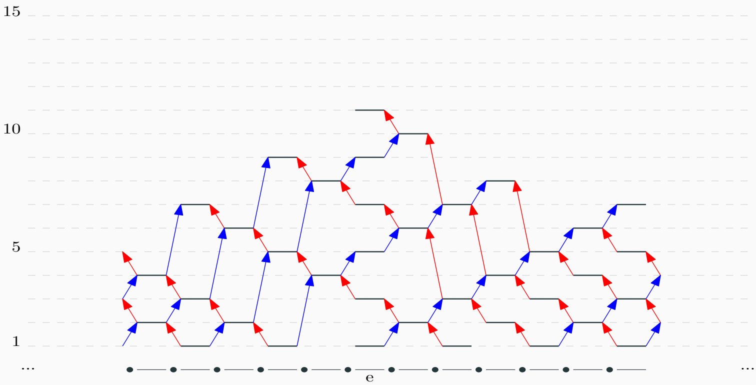

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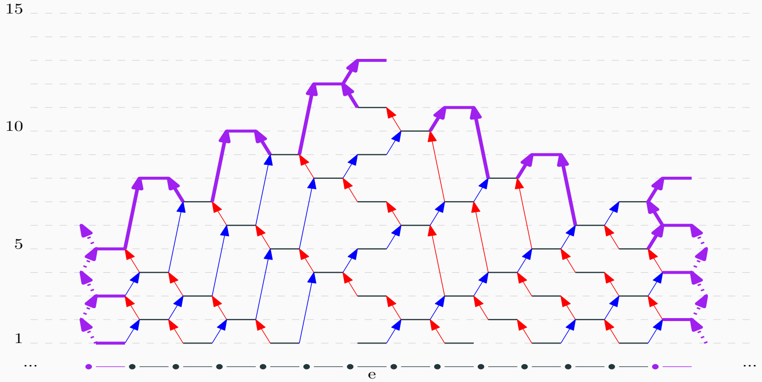

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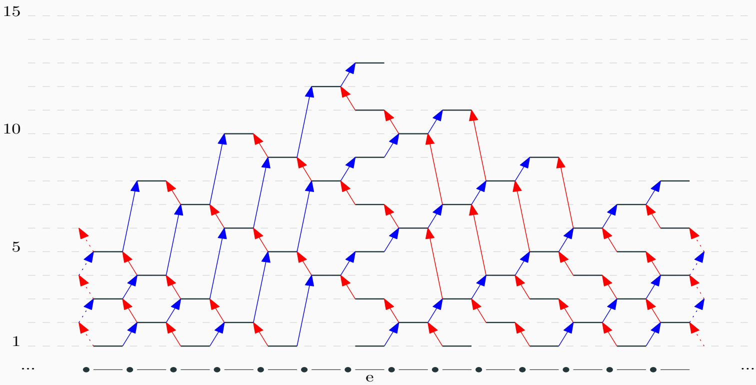

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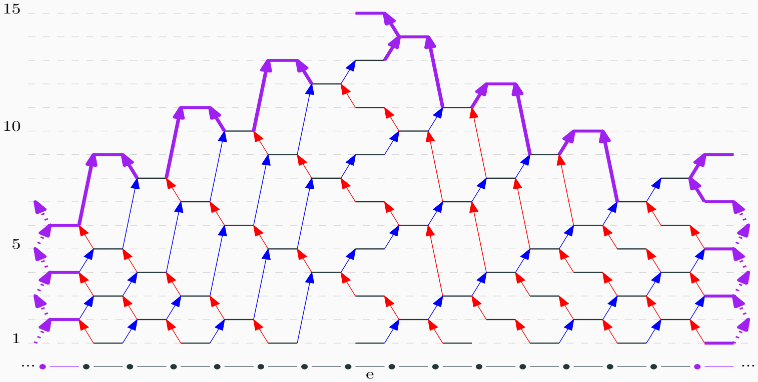

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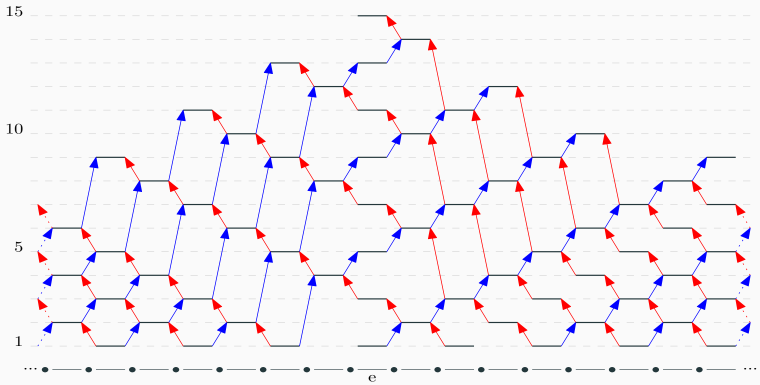


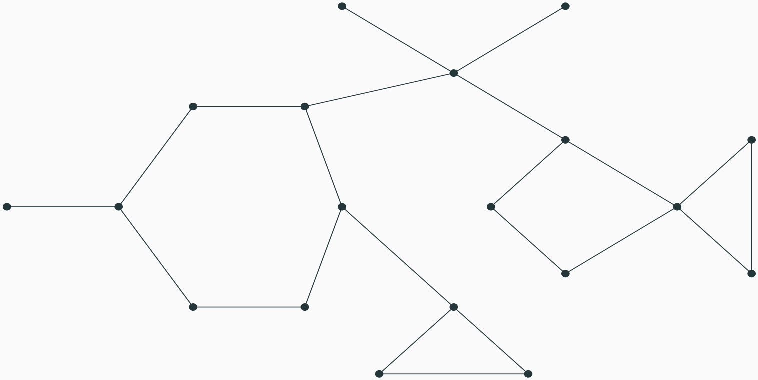
And finally: cactus graphs

Definition

A cactus graph is a tree graph with some vertices/edges replaced by cycles

Theorem

For any cactus graph on n vertices and circumference c (length of longest simple cycle), $\tau^+ \geq \frac{1}{2}c$, and $T^+ \geq \frac{1}{4}c^2 + 2(n-c) + 1$



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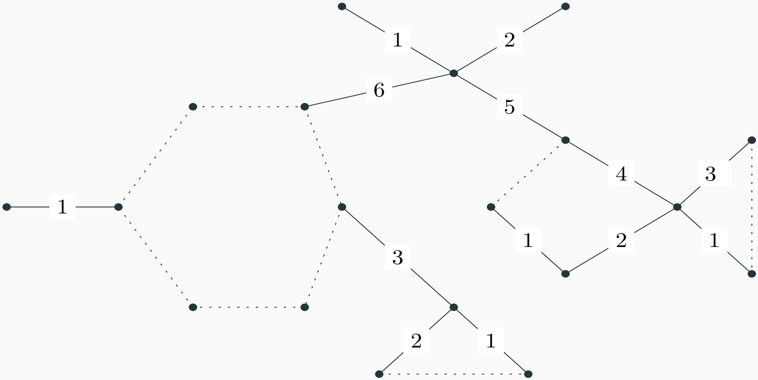

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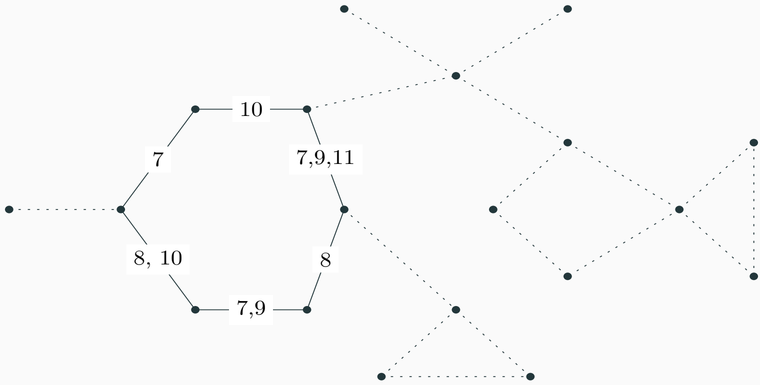

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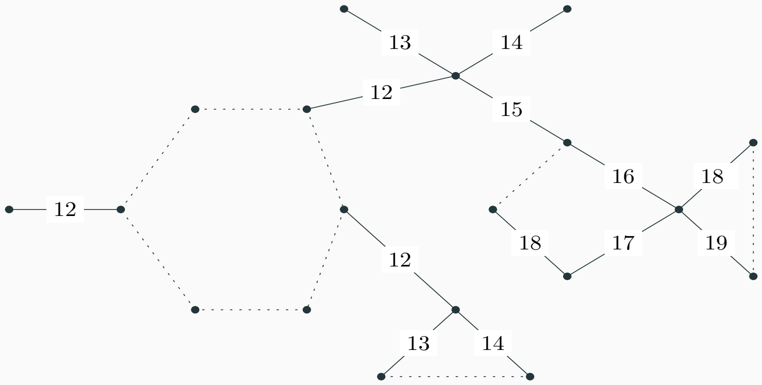

Conclusion

Theorem

For any tree graph on n vertices, $\tau^+ = 2$ and $T^+ = 2n - 3$

Corollary

For any connected graph on n vertices, $\tau^+ \geq 2$ and $T^+ \geq 2n - 3$

Theorem

For any cycle graph on n vertices, $\tau^+ \geq \frac{1}{2}n$ and $T^+ \geq \frac{1}{4}n^2 + 1$

Corollary

For any Hamiltonian graph on n vertices, $\tau^+ \geq \frac{1}{2}n$ and $T^+ \geq \frac{1}{4}n^2 + 1$

Theorem

For any cactus graph on n vertices and circumference c, $\tau^+ \geq \frac{1}{2}c$, and $T^+ \geq \frac{1}{4}c^2 + 2(n - c) + 1$

Corollary

For any graph on n vertices and circumference c, $\tau^+ \geq \frac{1}{2}c$, and $T^+ \geq \frac{1}{4}c^2 + 2(n - c) + 1$

Conclusion

Future work and open questions

- Reduce upper bound of τ^+ in cycles, currently $n-1$ (to a tight $\frac{1}{2}n$?) by showing forbidden configurations of dominating journeys
- Can cactus graphs beat $\tau^+ = \frac{1}{2}c$?
- We adapted our labelling generator to work for general graphs, and $\tau^+ \leq \frac{1}{2}n$ (empirically) seems to hold in general as well
- Given a general graph, is it NP-hard to compute τ^+ or T^+?

Measure \ Labelling	Proper	Happy	Strict
τ^+	$\geq \frac{1}{2}n$	1 (by definition)	$> \frac{1}{2}n$? (empirically)
T^+	$\geq \frac{1}{4}n^2+1$	$\geq \frac{1}{18}n^2+O(n)$	$\geq \frac{1}{2}n^2+O(n)$

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Thank you for your attention

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