Sparse Temporal Spanners with Low Stretch

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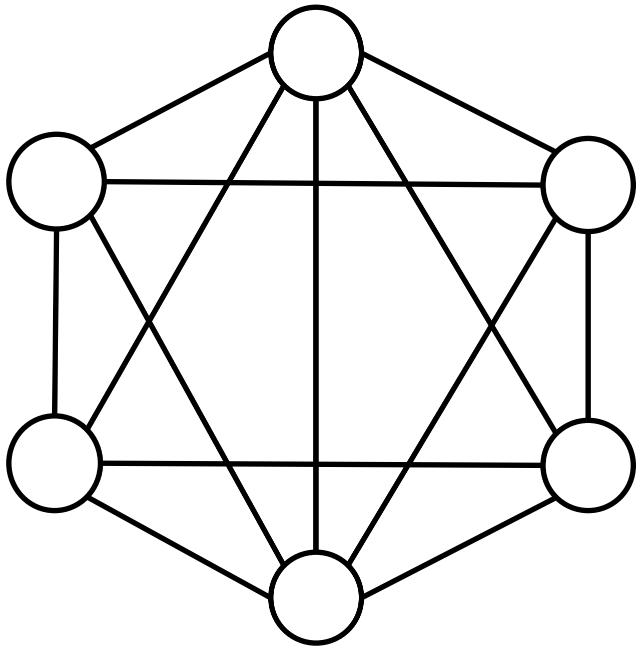
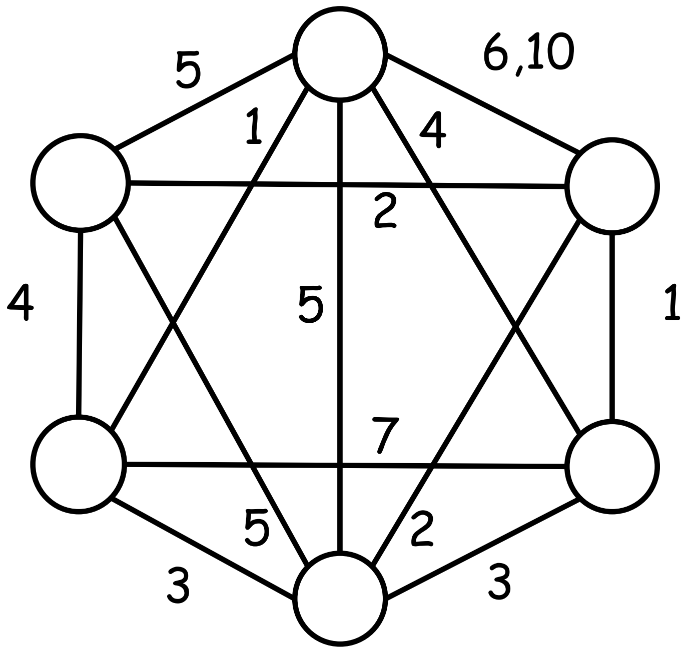
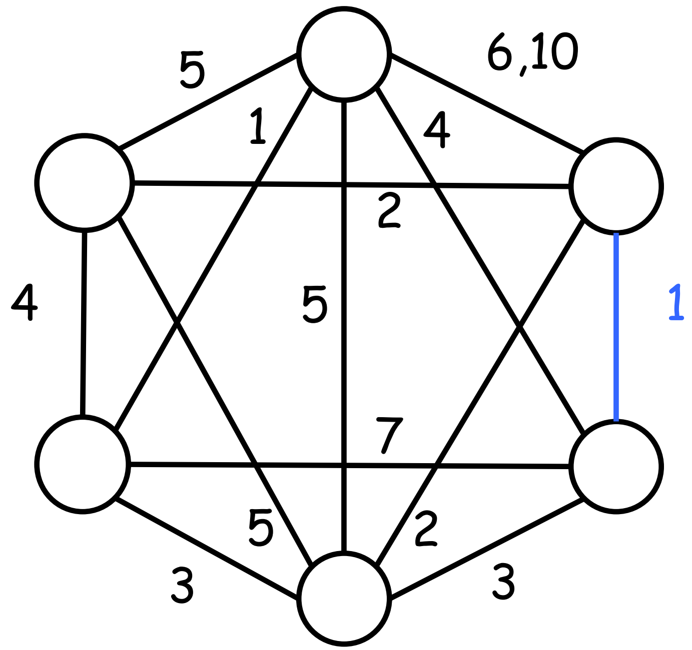
Temporal graphs



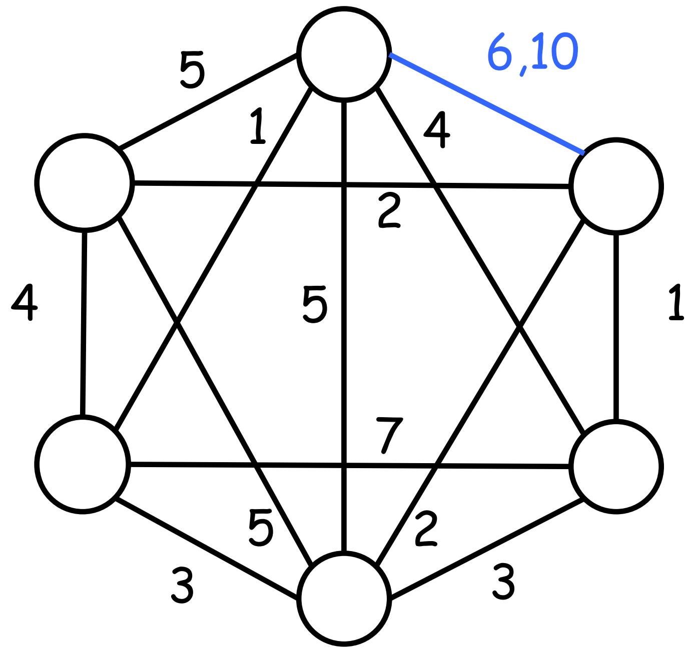
Temporal graphs



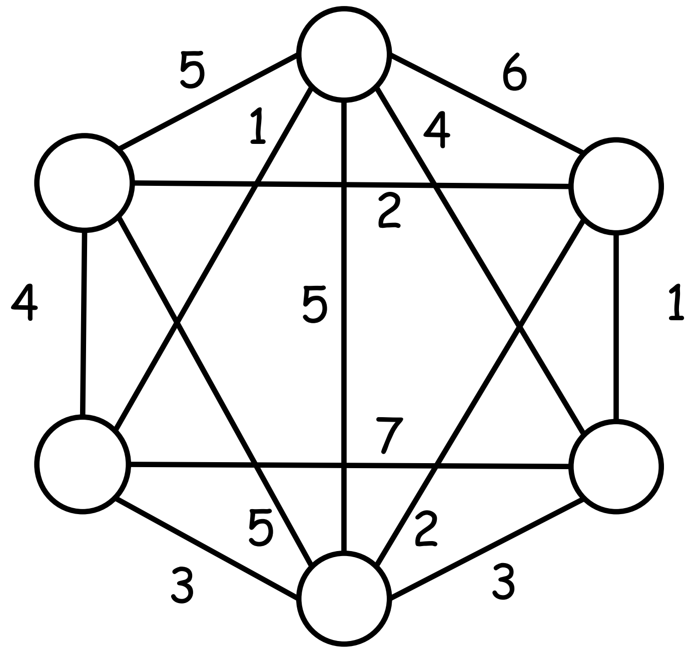
Temporal graphs



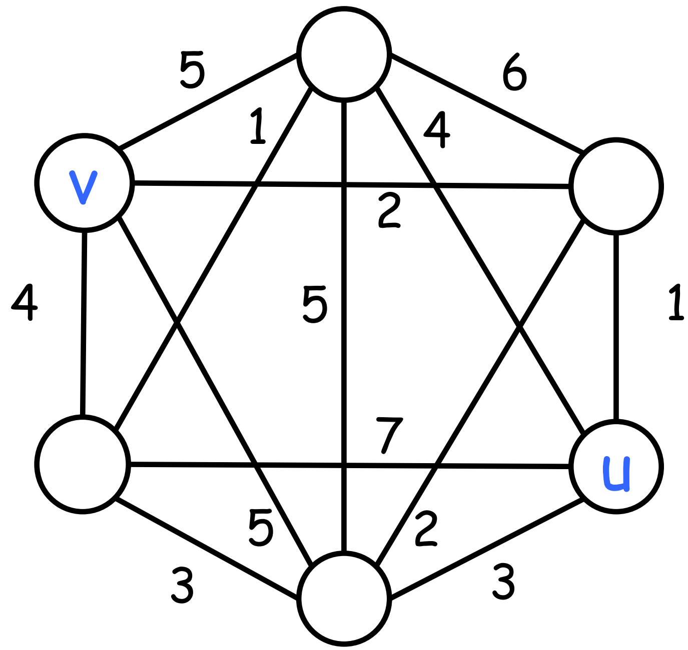
Temporal graphs



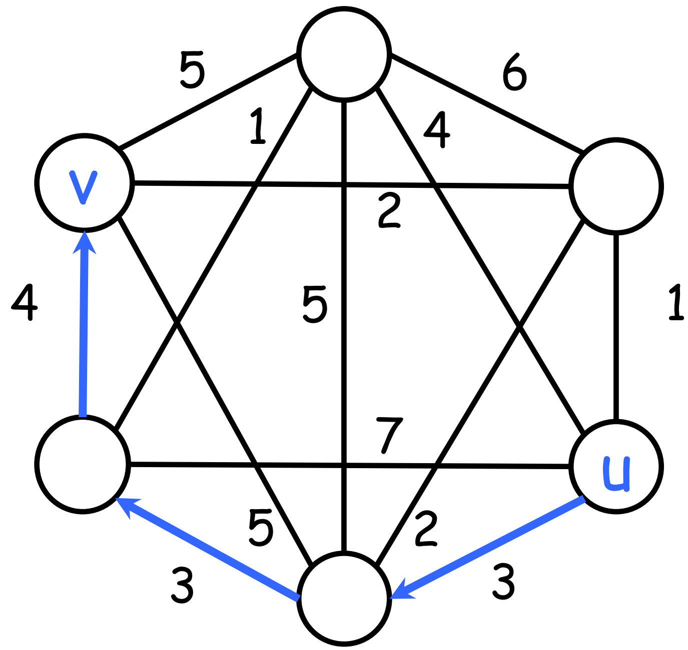
Temporal graphs



Temporal graphs



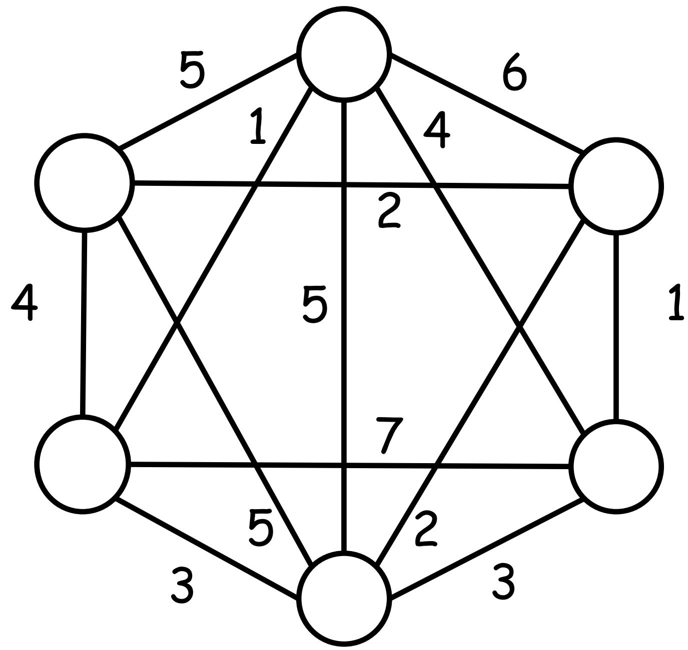
Temporal graphs



u-v temporal path: u-v path of non-decreasing time-labels

Temporal spanner:

a subgraph H of G that preserves pairwise temporal connectivity



Temporal spanner:

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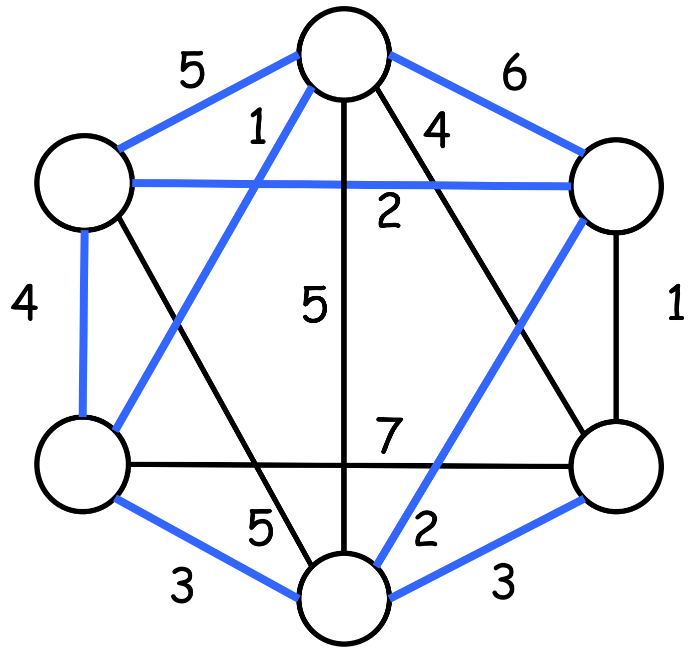
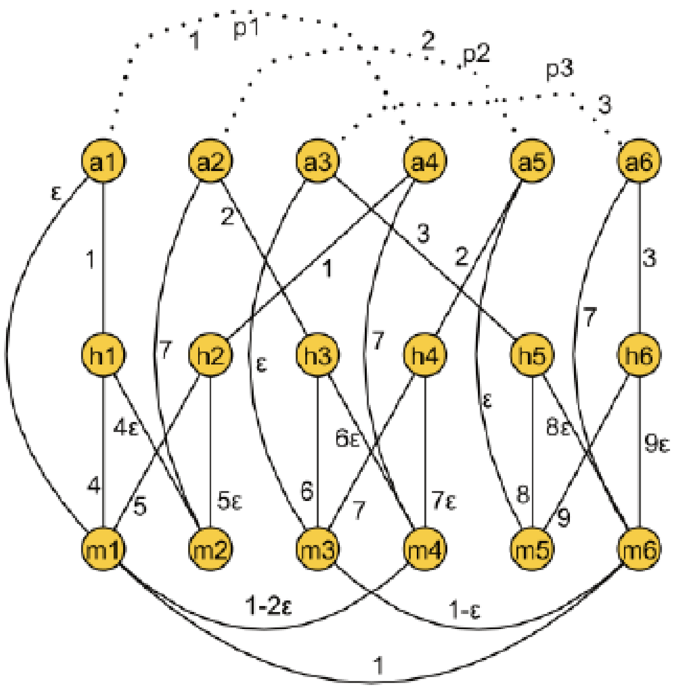


Kempe et al. [STOC'00]: find temporal spanners of small size (#of edges)

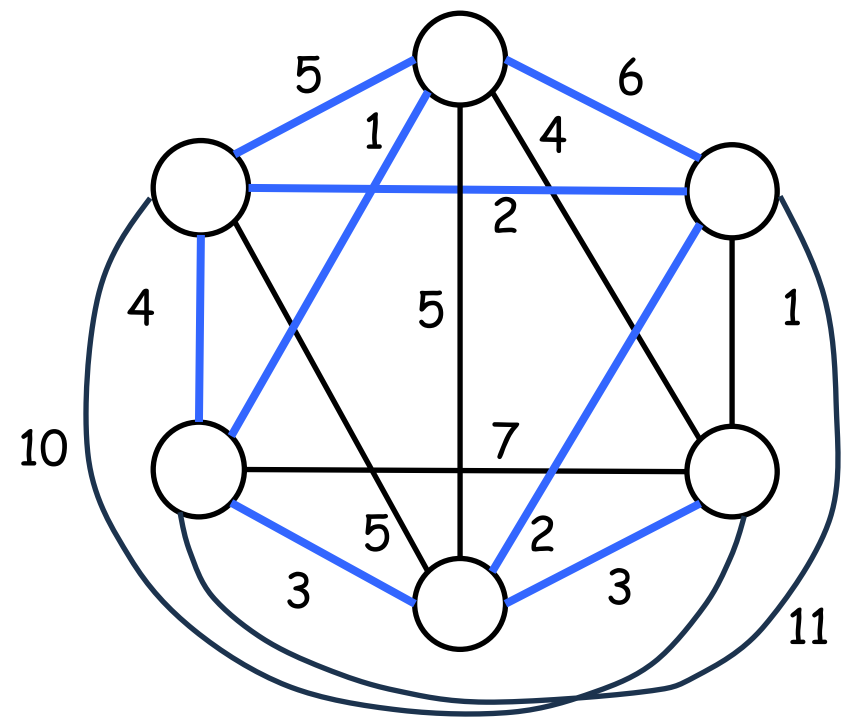
Axiotis and Fotakis
ICALP 2016

Lower Bound of $\Omega(n^2)$



Casteigts et al.
ICALP 2019

Upper Bound of O(n log n)
for temporal cliques

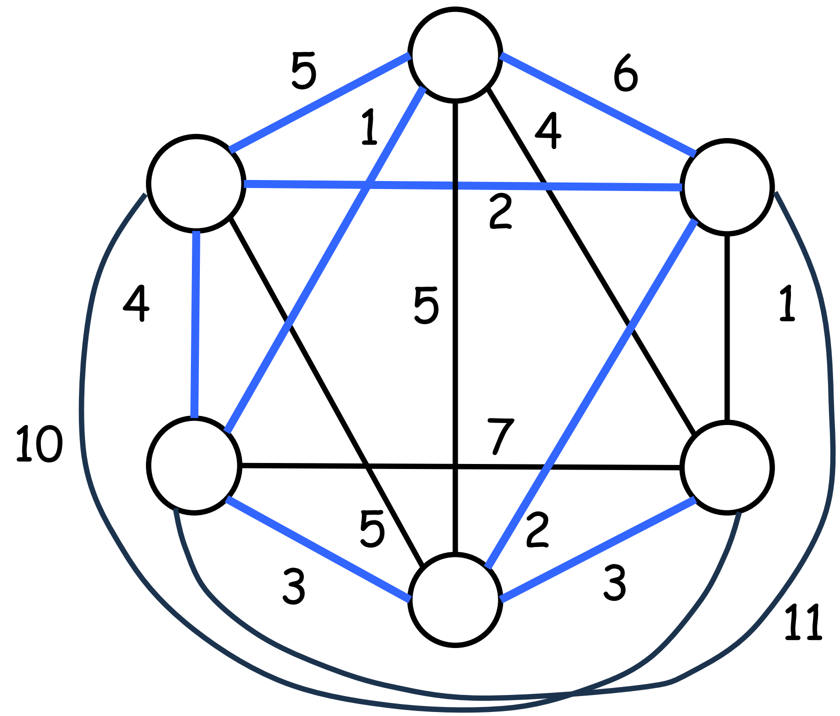


only preserves reachability

no guarantees on the distances

Casteigts et al.
ICALP 2019

Upper Bound of $O(n \log n)$ for temporal cliques



Temporal spanner with stretch α:

a subgraph H of G such that for every pair of vertices u and v

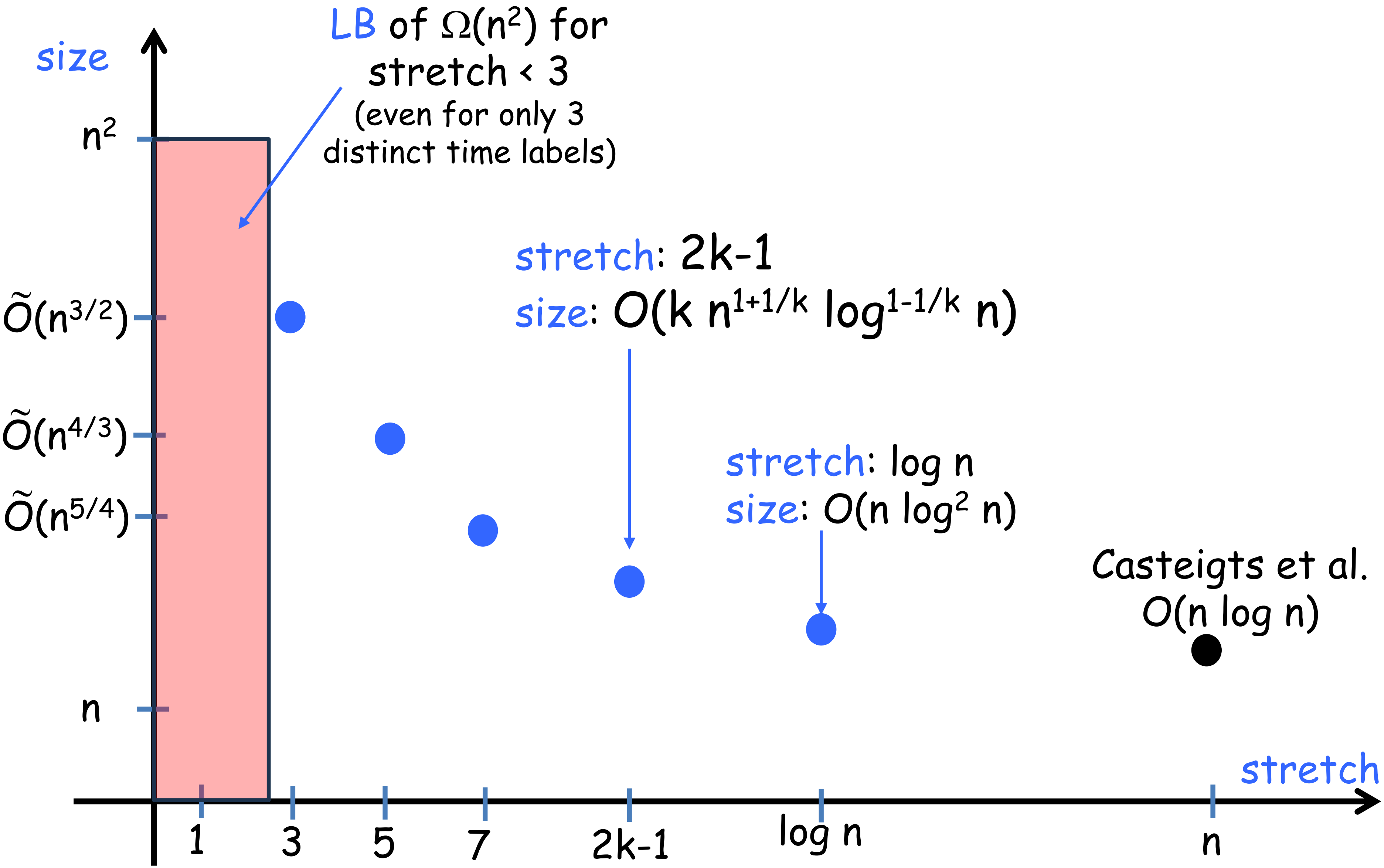
$$dist_H(u,v) \leq \alpha \, dist_G(u,v)$$

distance:

1. min Length (#of edges) — generalization of standard notion of distance for static graphs
2. min Arrival Time
3. max Departure Time
4. min Travel time

(2–4: strong lower bounds)

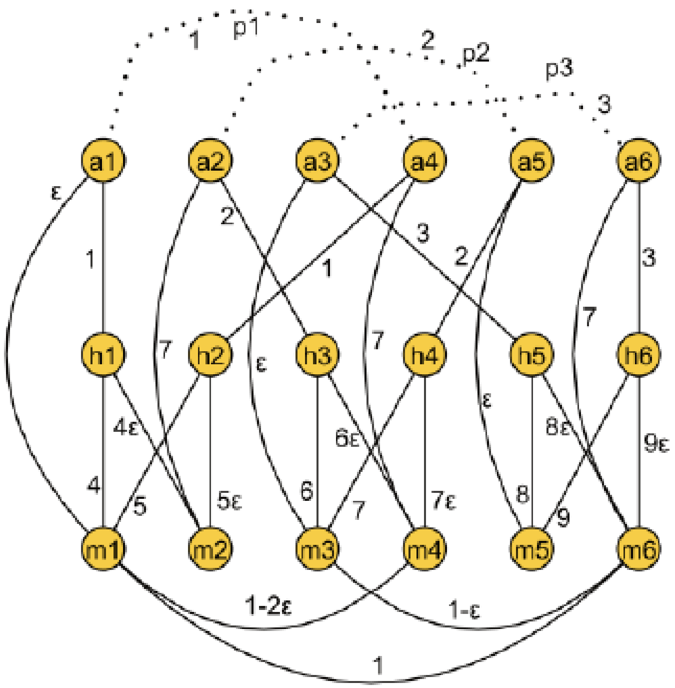
Our results I: cliques



Our results II: general graphs

Axiotis and Fotakis
ICALP 2016

Lower Bound of $\Omega(n^2)$



Our results II: single-source spanners for general graphs

a subgraph H of G such that for every vertex v

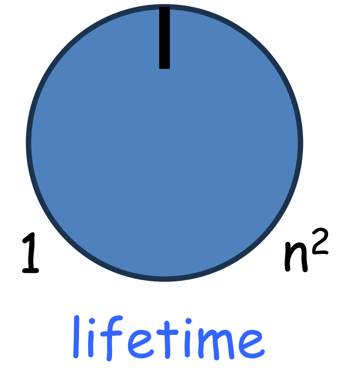
$$\mathrm{dist}_H(s,v) \leq \alpha\, \mathrm{dist}_G(s,v)$$

UB: stretch: $1+\varepsilon$ size: $O\left(n \frac{\log^4 n}{\log(1+\varepsilon)}\right)$

LB: size $\Omega(n^2)$ for stretch 1

Our results III: the role of lifetime

lifetime: number L of distinct time-labels



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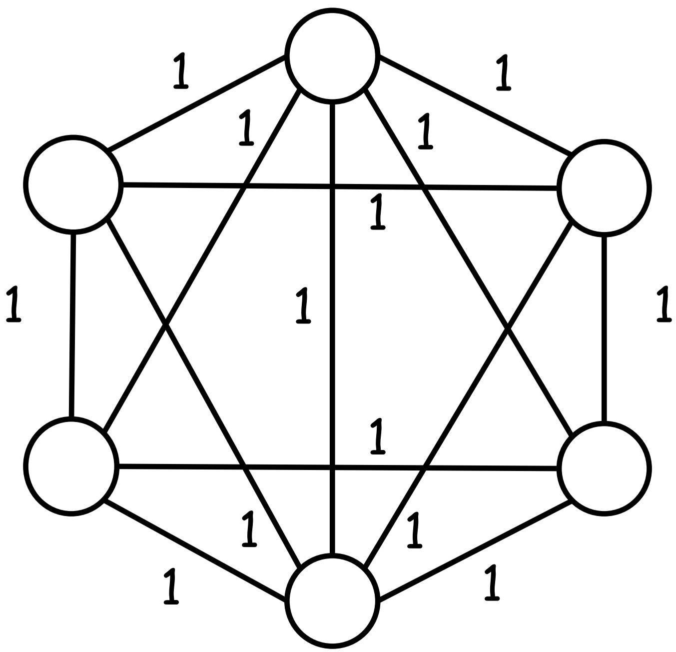

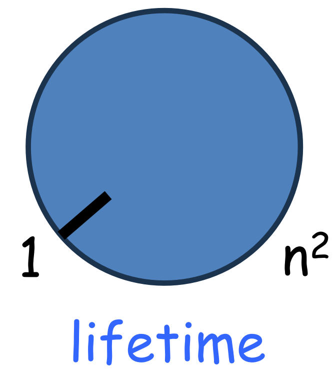

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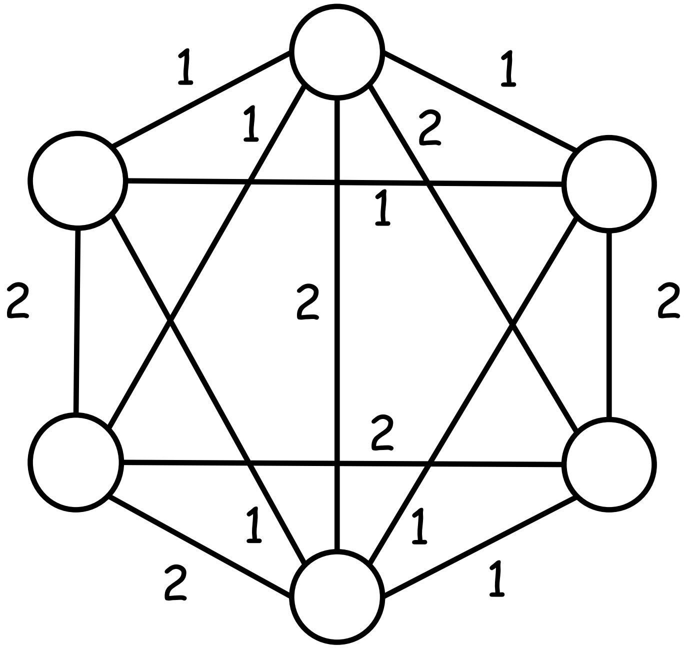

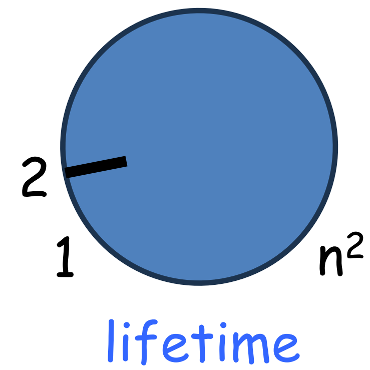

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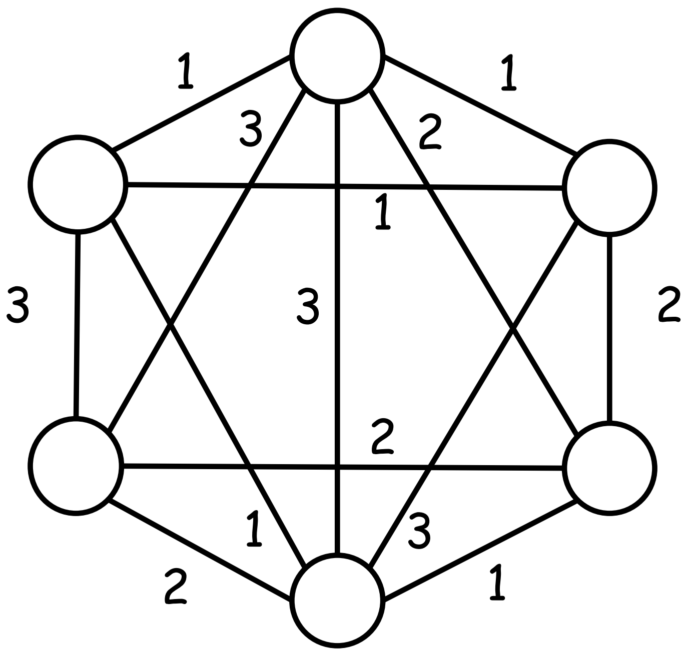
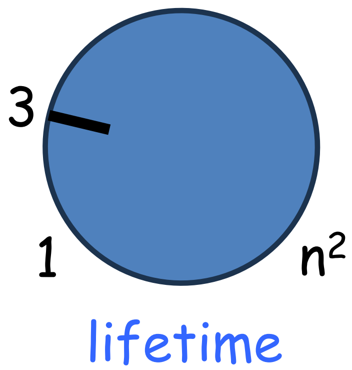


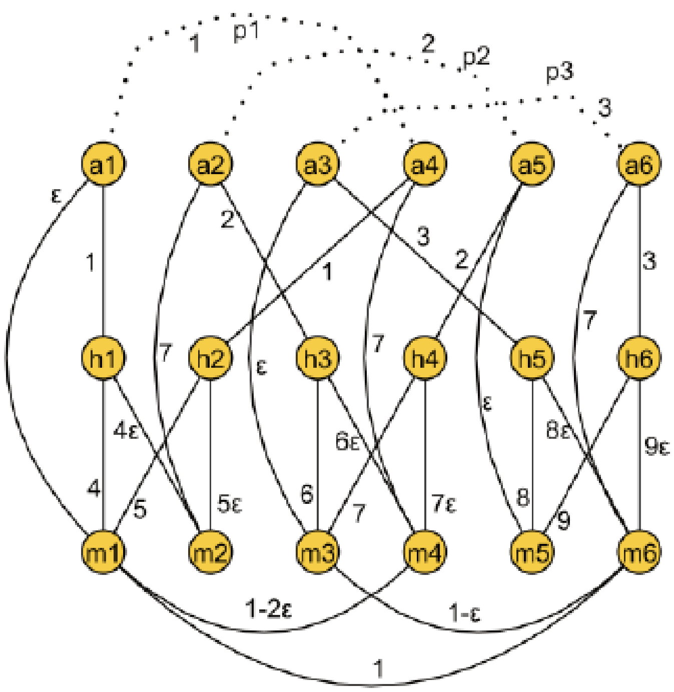


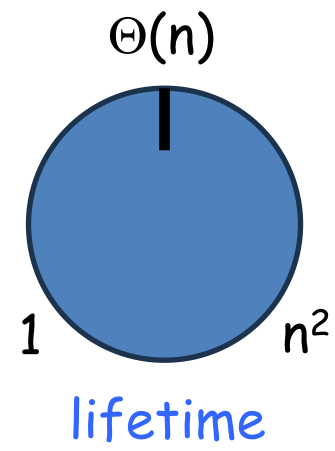
Our results III: the role of lifetime

lifetime: number L of distinct time-labels

Axiotis and Fotakis
ICALP 2016

Lower Bound of $\Omega(n^2)$



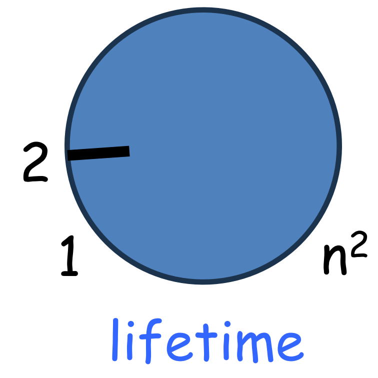


Our results III: the role of lifetime

Cliques

α=2

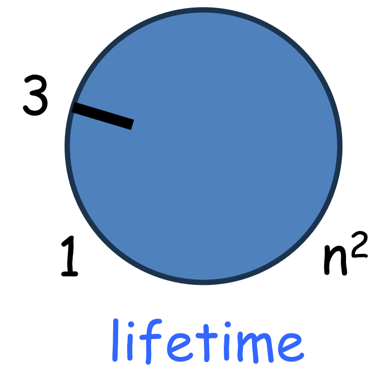
L=2 O(n log n)



Our results III: the role of lifetime

Cliques

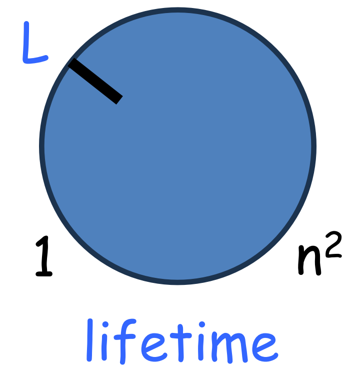
	α=2	α=3
L=2	$O(n \log n)$	
L=3	$\Omega(n^2)$	$O(n \log n)$



Our results III: the role of lifetime

Cliques

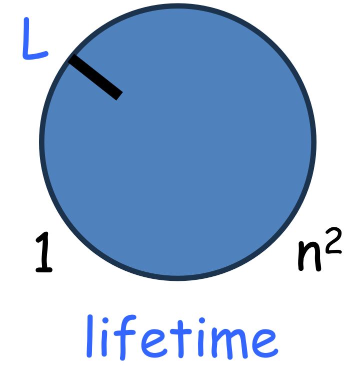
	α=2	α=3
L=2	O(n log n)	
L=3	$\Omega(n^2)$	O(n log n)
L		$O(2^L n \log n)$



Our results III: the role of lifetime

Cliques

	$\alpha=2$	$\alpha=3$
$L=2$	$O(n \log n)$	
$L=3$	$\Omega(n^2)$	$O(n \log n)$
L		$O(2^L n \log n)$



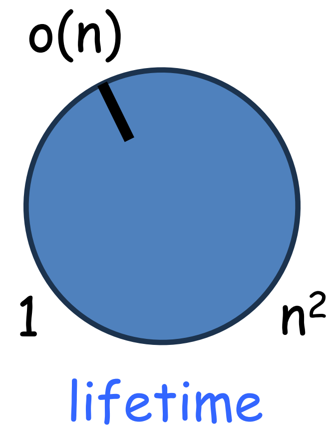
General graphs

an α-spanner of size $f(n)$ for static graphs → an α-spanner of size $O(L f(n))$ for temporal graphs of lifetime L

Our results III: the role of lifetime

Cliques

	$\alpha=2$	$\alpha=3$
$L=2$	$O(n \log n)$	
$L=3$	$\Omega(n^2)$	$O(n \log n)$
L		$O(2^L n \log n)$



General graphs

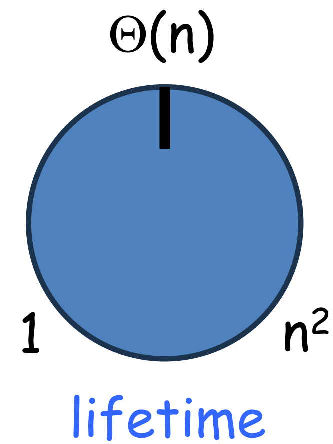
an α-spanner of size $f(n)$ for static graphs → an α-spanner of size $O(L f(n))$ for temporal graphs of lifetime L

⇒ temporal spanner of stretch $\log n$ and size $o(n^2)$ for any temporal graph o lifetime $L=o(n)$

Our results III: the role of lifetime

Cliques

	$\alpha=2$	$\alpha=3$
$L=2$	$O(n \log n)$	
$L=3$	$\Omega(n^2)$	$O(n \log n)$
L		$O(2^L n \log n)$



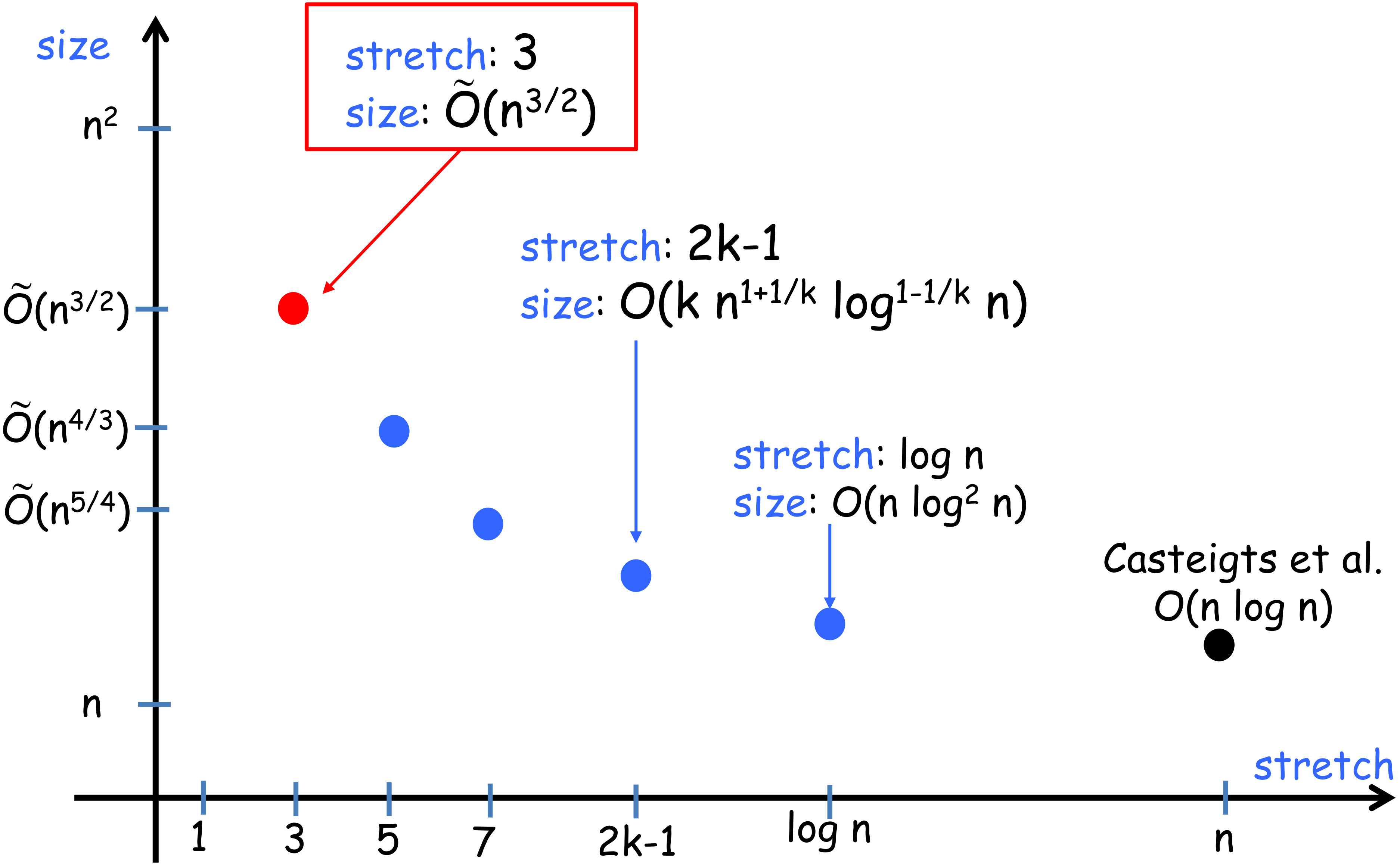
General graphs

an α-spanner of size $f(n)$ for static graphs → an α-spanner of size $O(Lf(n))$ for temporal graphs of lifetime L

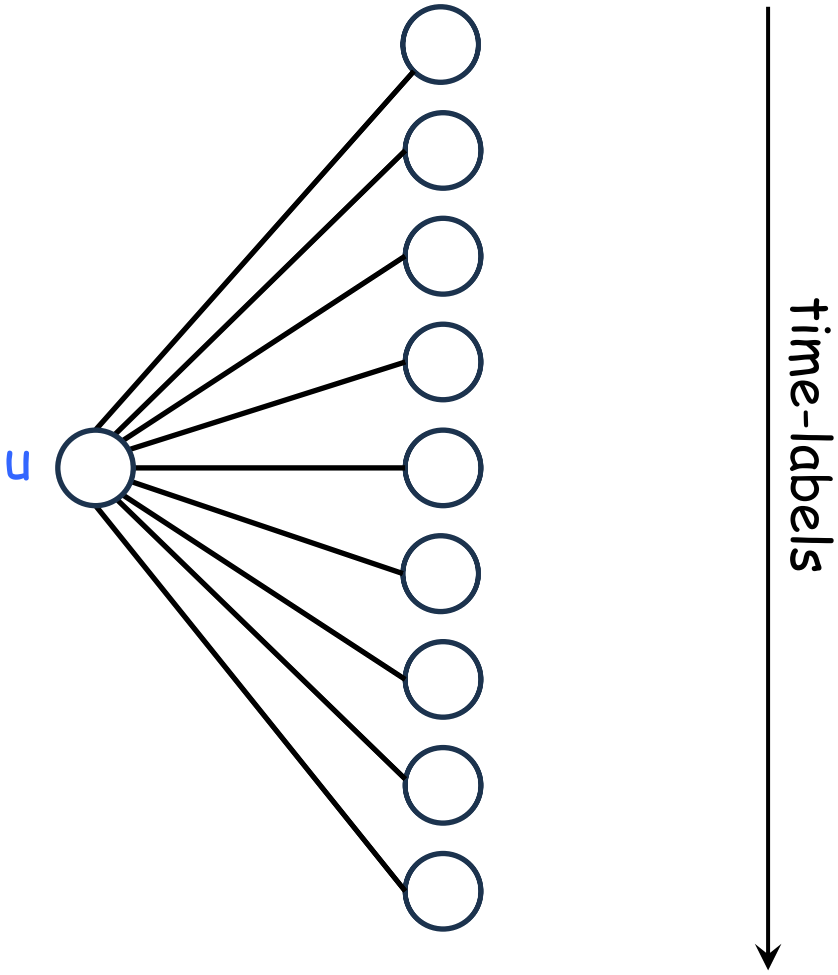
⇒ temporal spanner of stretch $\log n$ and size $o(n^2)$ for any temporal graph o lifetime $L=o(n)$

size $\Omega(n^2)$ for general graph with $L=\Theta(n)$

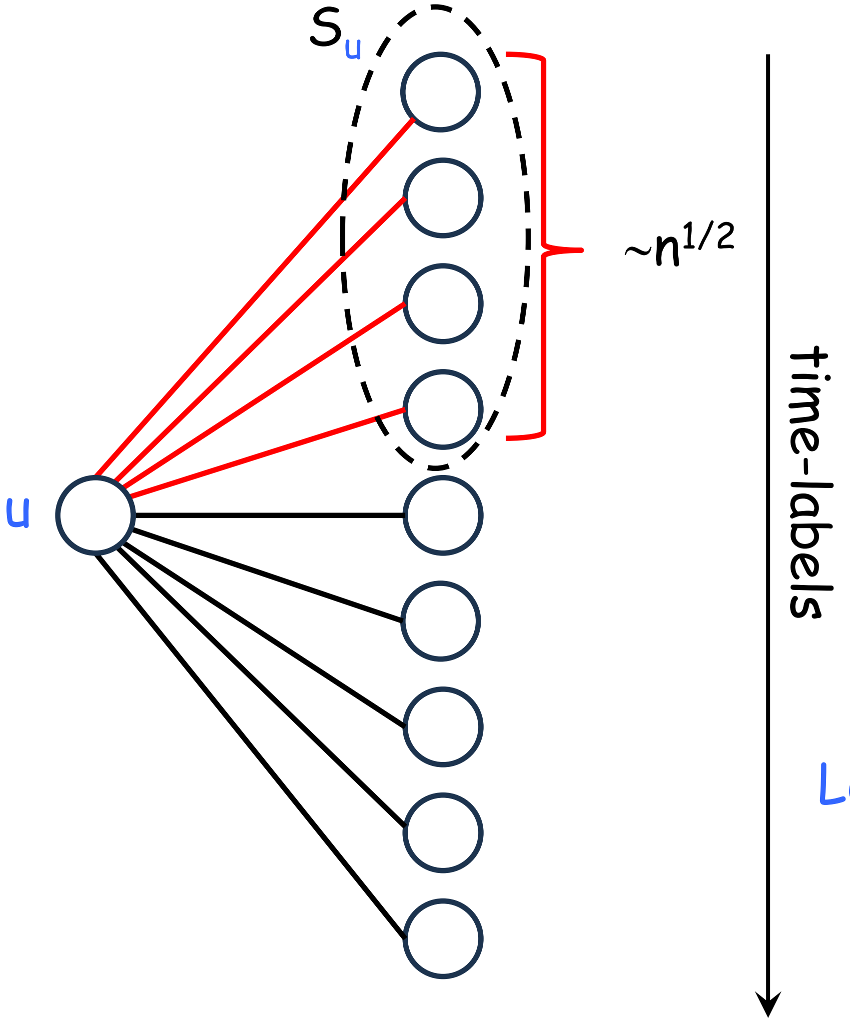
Our results I: cliques



for every u∈V



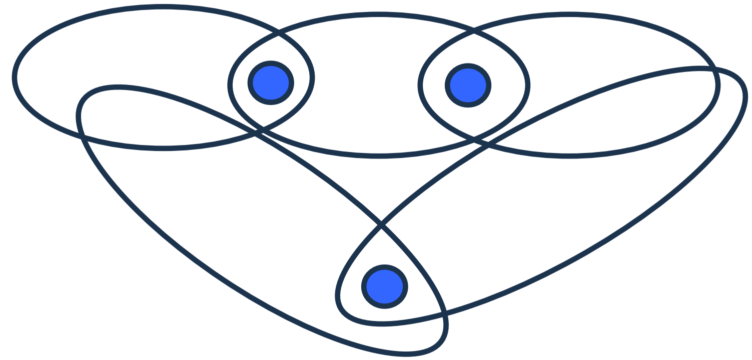
for every $u \in V$



H:= red edges

- # red edges $\tilde{O}(n^{3/2})$

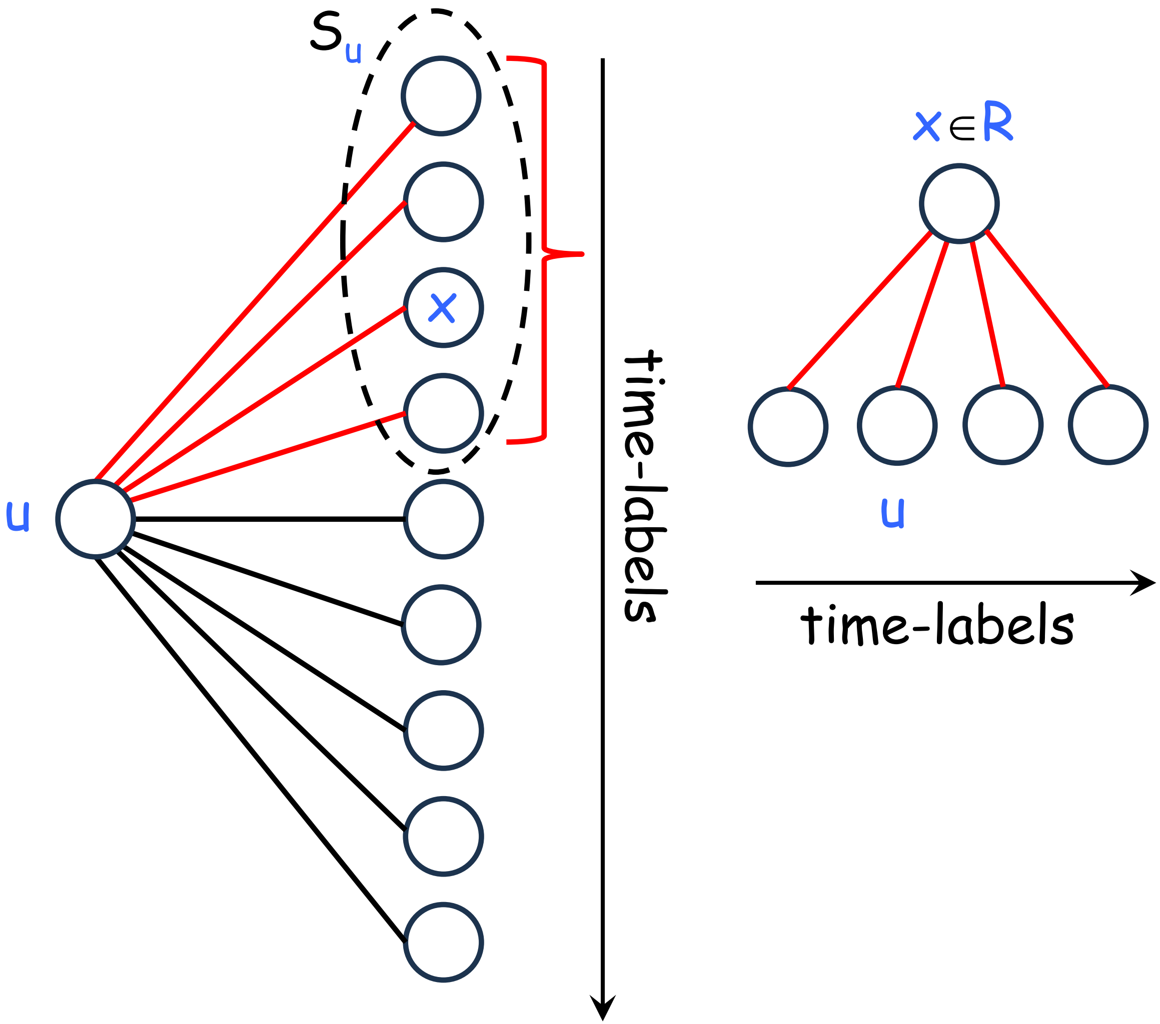
compute a hitting set R of S_u's



Lemma: $|R| = \tilde{O}\left(\frac{n}{|S_u|}\right) = \tilde{O}\left(\frac{n}{n^{1/2}}\right) = \tilde{O}(n^{1/2})$

for every u∈V

cluster the vertices around R



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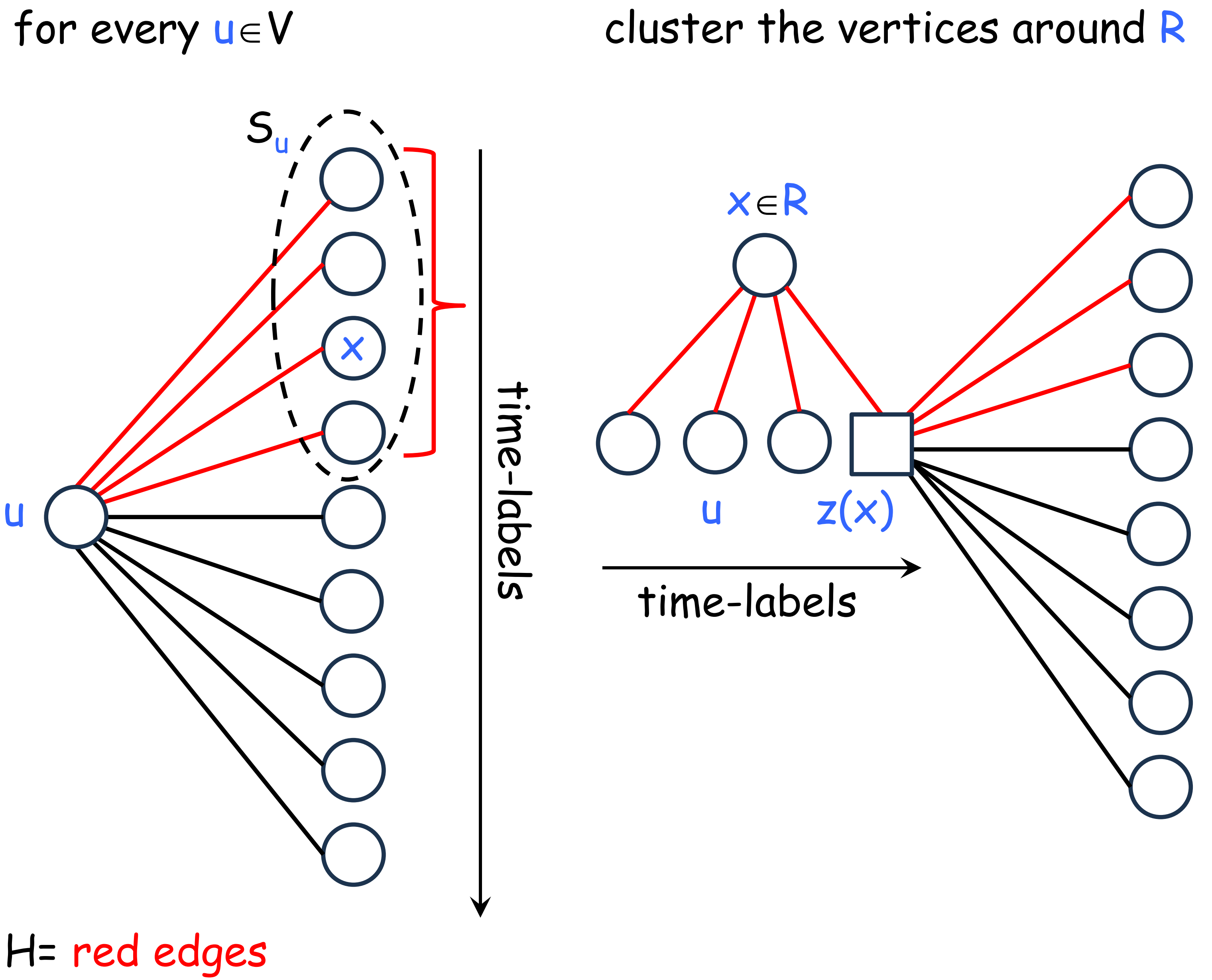


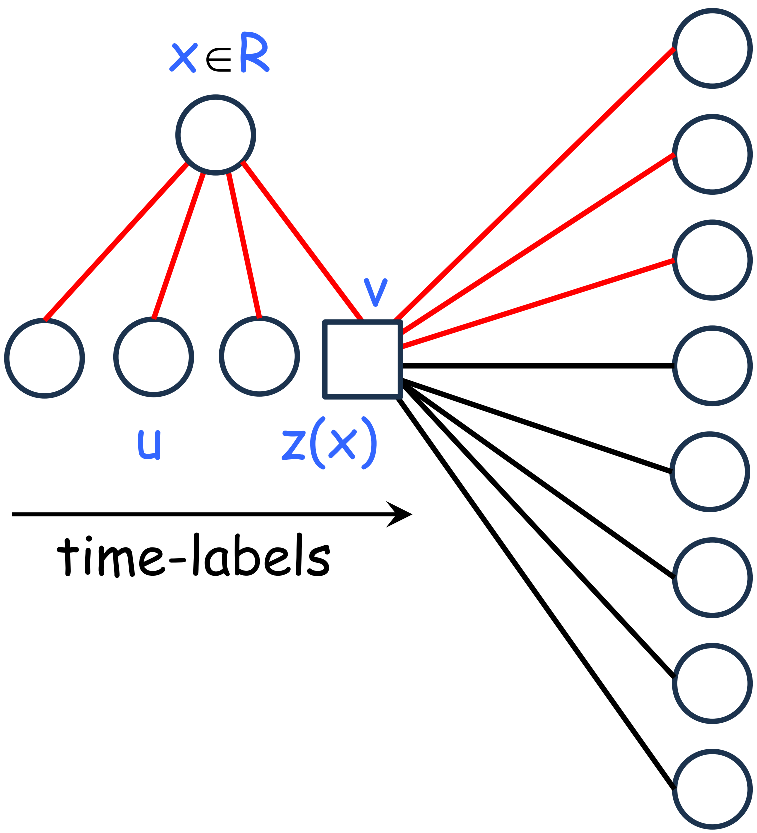
for every u∈V
cluster the vertices around R
S_u
x
u
time-labels
x∈R
u
z(x)
time-labels
H= red edges

for every u∈V

cluster the vertices around R

for any v∈V

case: v = z(x)



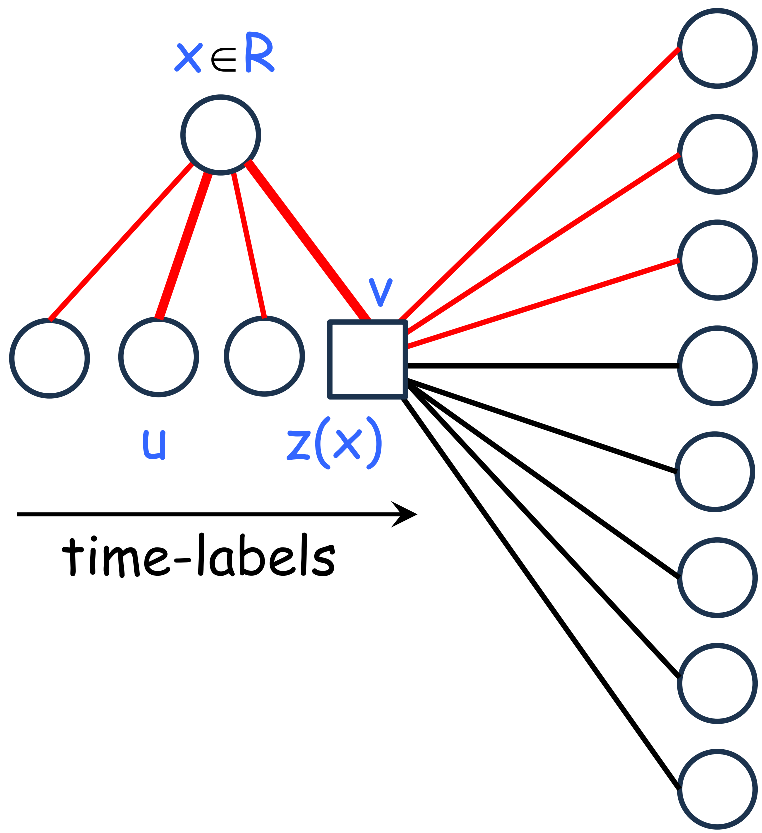
H= red edges

for every $u \in V$

cluster the vertices around R

for any $v \in V$

case: $v = z(x)$



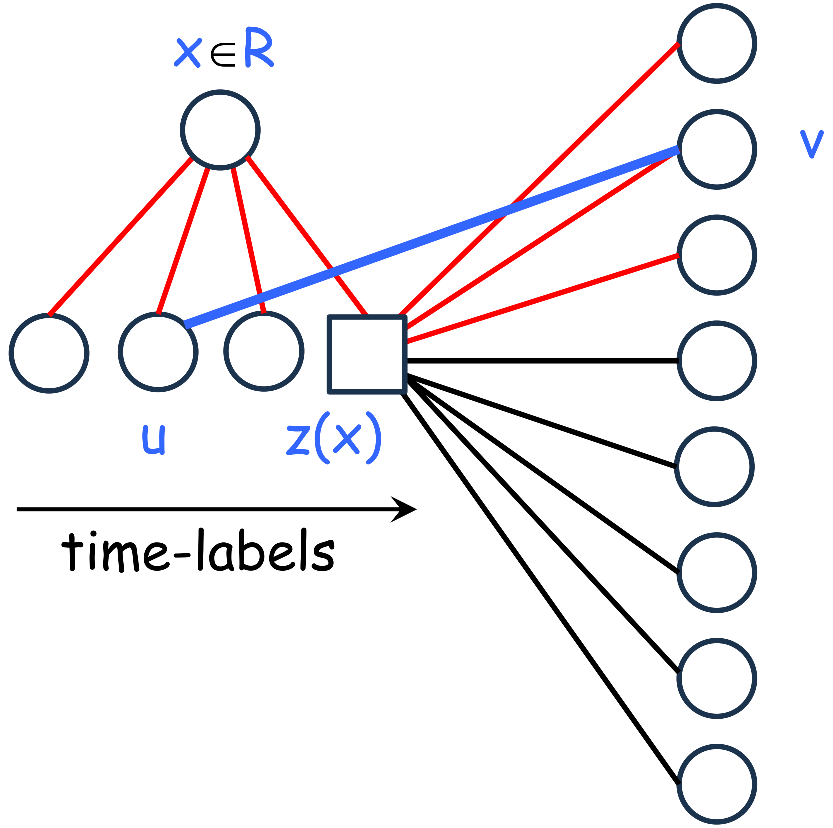
H= red edges

for every $u \in V$

cluster the vertices around R

for any $v \in V$

case: $v \in S_{z(x)}$



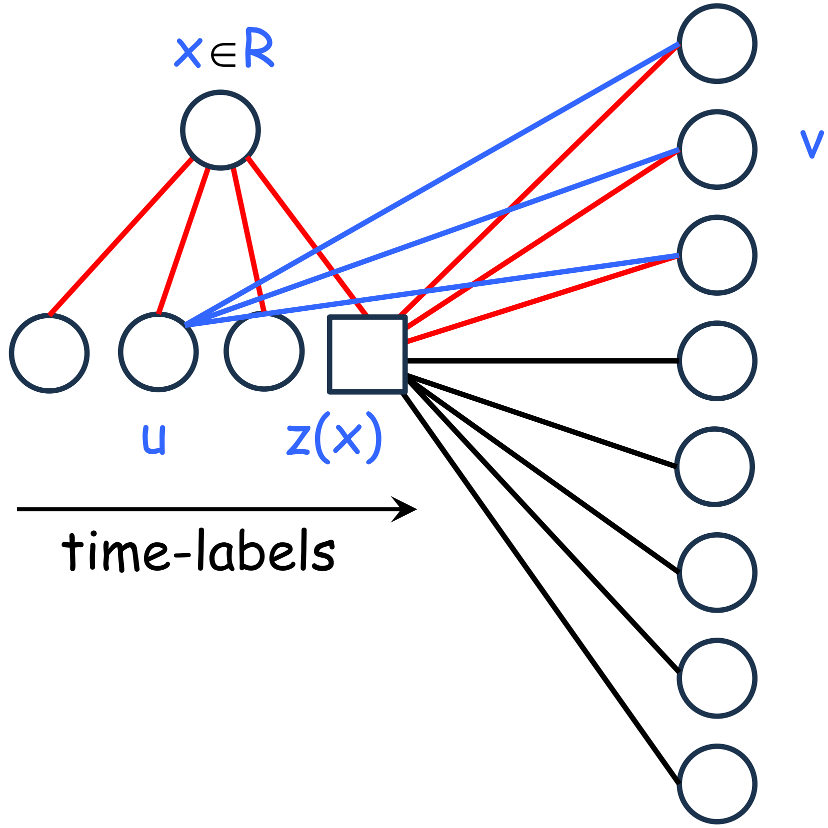
H= red edges

for every $u \in V$

cluster the vertices around R

for any $v \in V$

case: $v \in S_{z(x)}$



H= red edges + blue edges

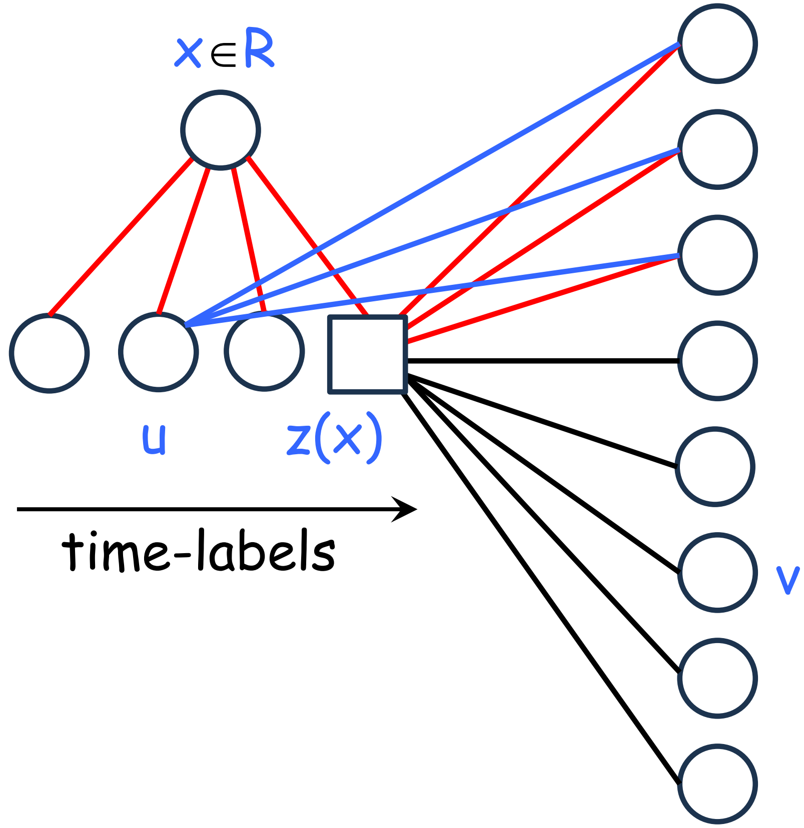
- # blue edges $\tilde{O}(n^{3/2})$

for every $u \in V$

cluster the vertices around R

for any $v \in V$

case: $v \in V \setminus S_{z(x)}$



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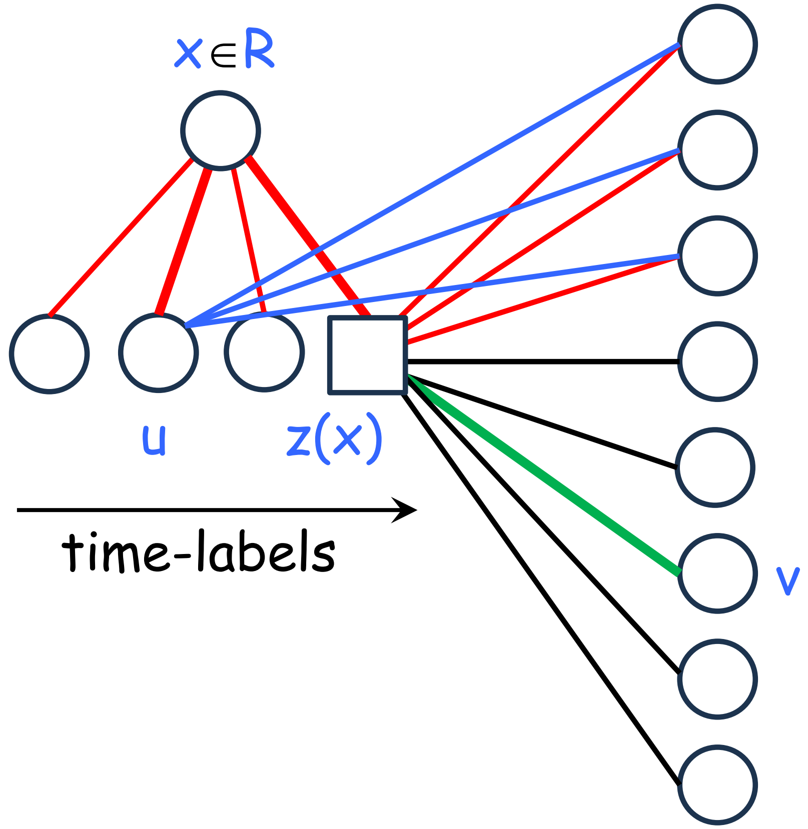


H= red edges + blue edges

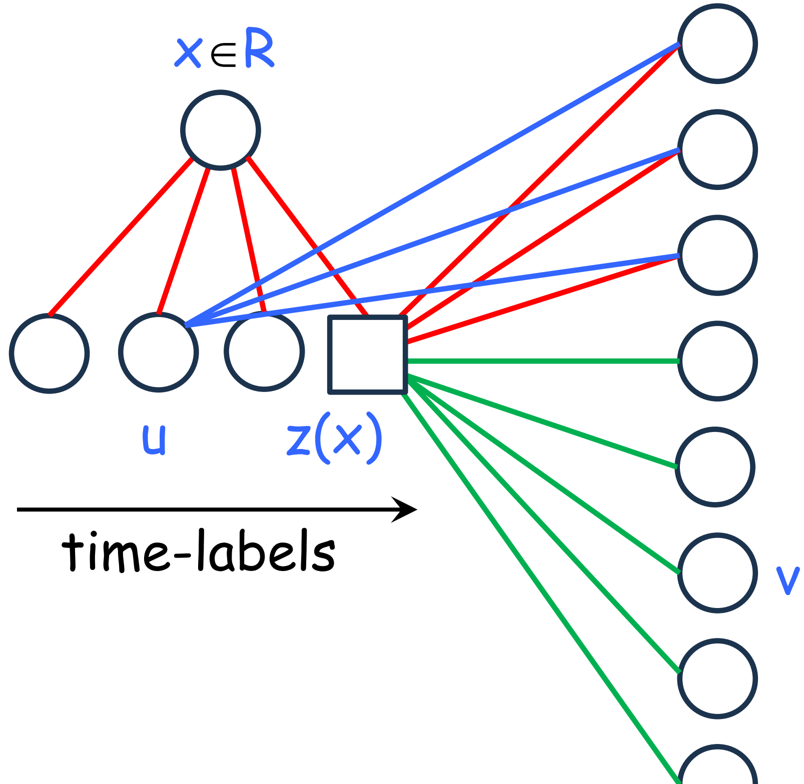
- # blue edges $\tilde{O}(n^{3/2})$

for every $u \in V$

cluster the vertices around R

for any $v \in V$

case: $v \in V \setminus S_{z(x)}$

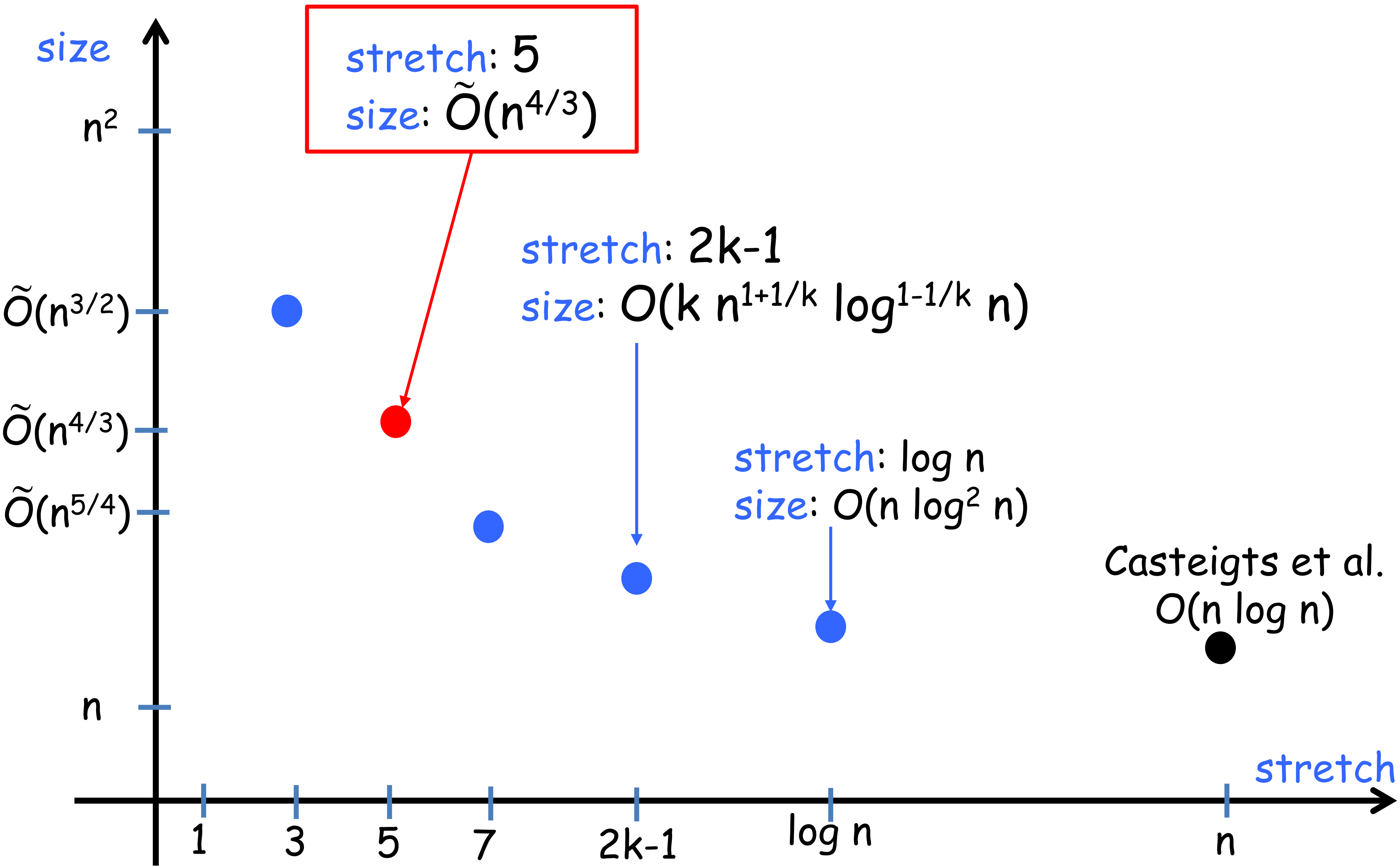


H is a 3- spanner of size $\tilde{O}(n^{3/2})$

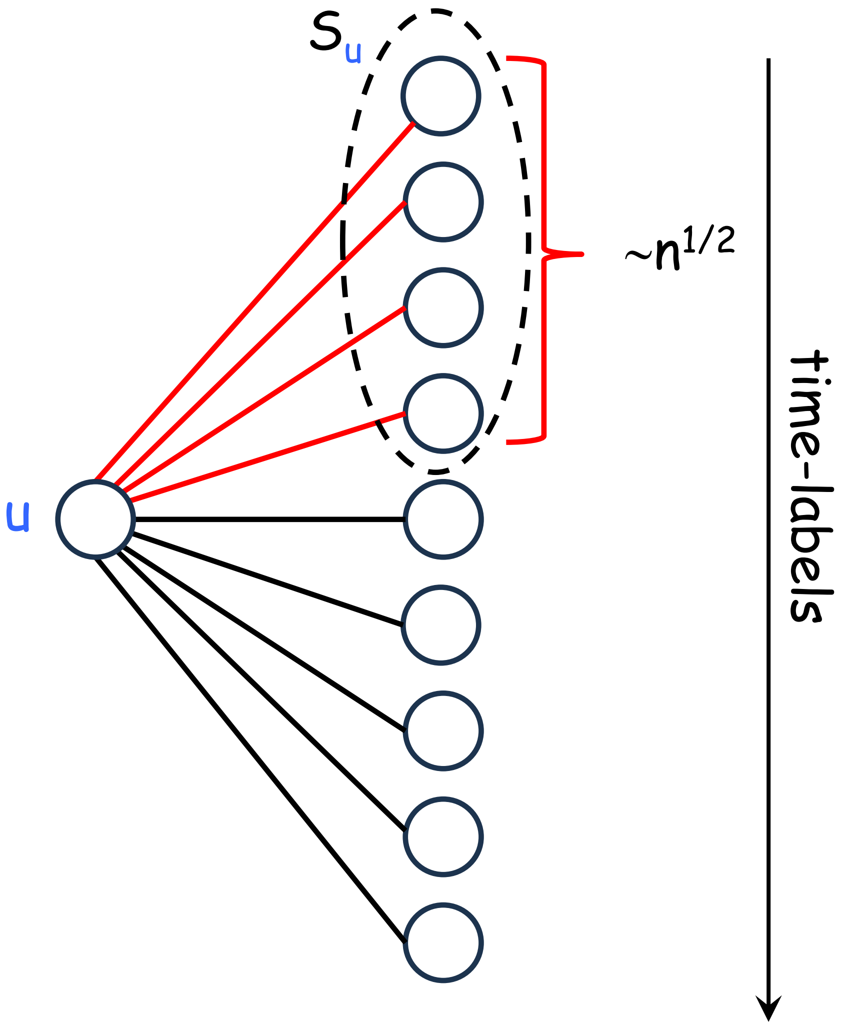
H= red edges + blue edges + green edges

- # green edges $\tilde{O}(n^{3/2})$

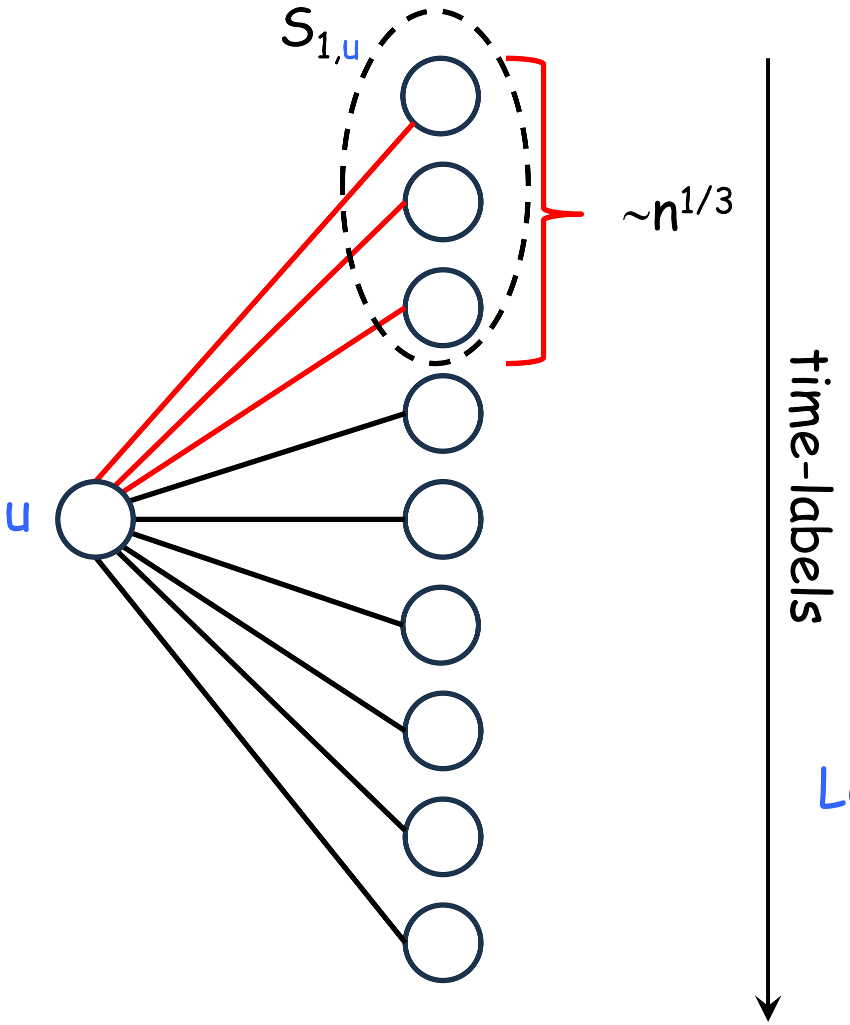
Our results I: cliques



for every $u \in V$



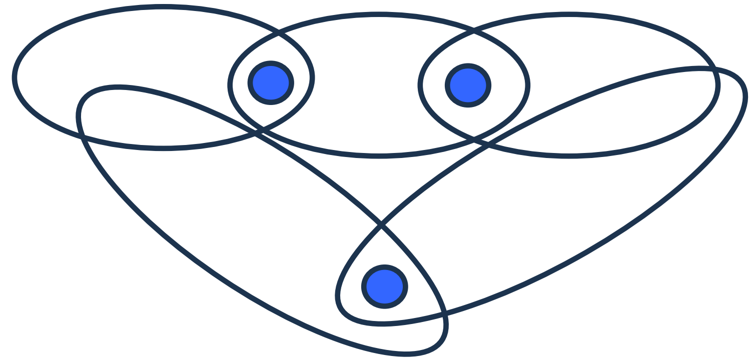
for every $u \in V$



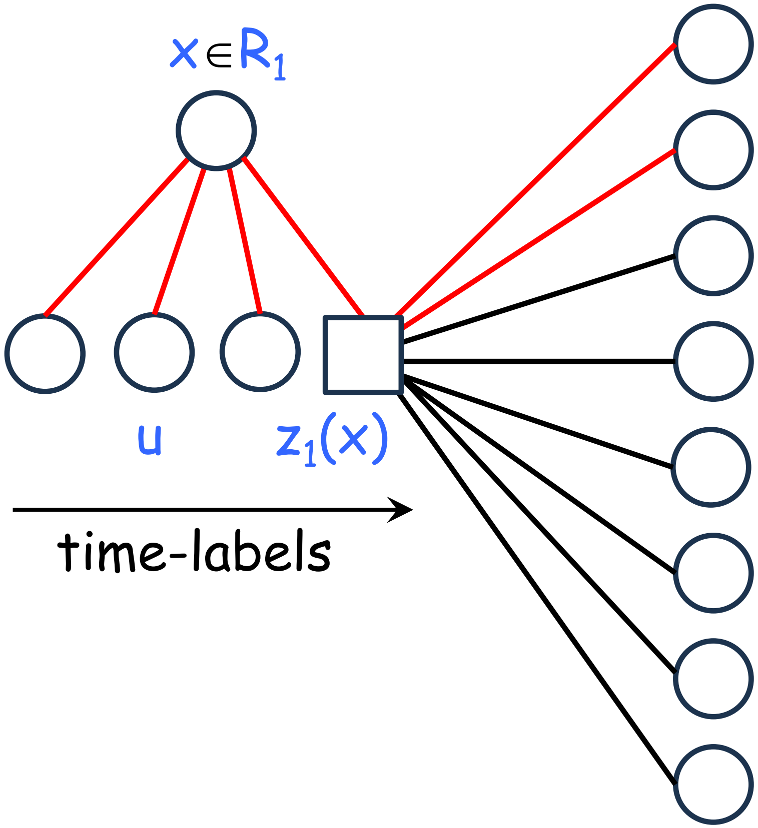
$H :=$ red edges

- # red edges $\tilde{O}(n^{4/3})$

compute a hitting set R_1 of $S_{1,u}$'s



Lemma: $|R_1| = \tilde{O}\left(\frac{n}{|S_{1,u}|}\right) = \tilde{O}\left(\frac{n}{n^{1/3}}\right) = \tilde{O}(n^{2/3})$

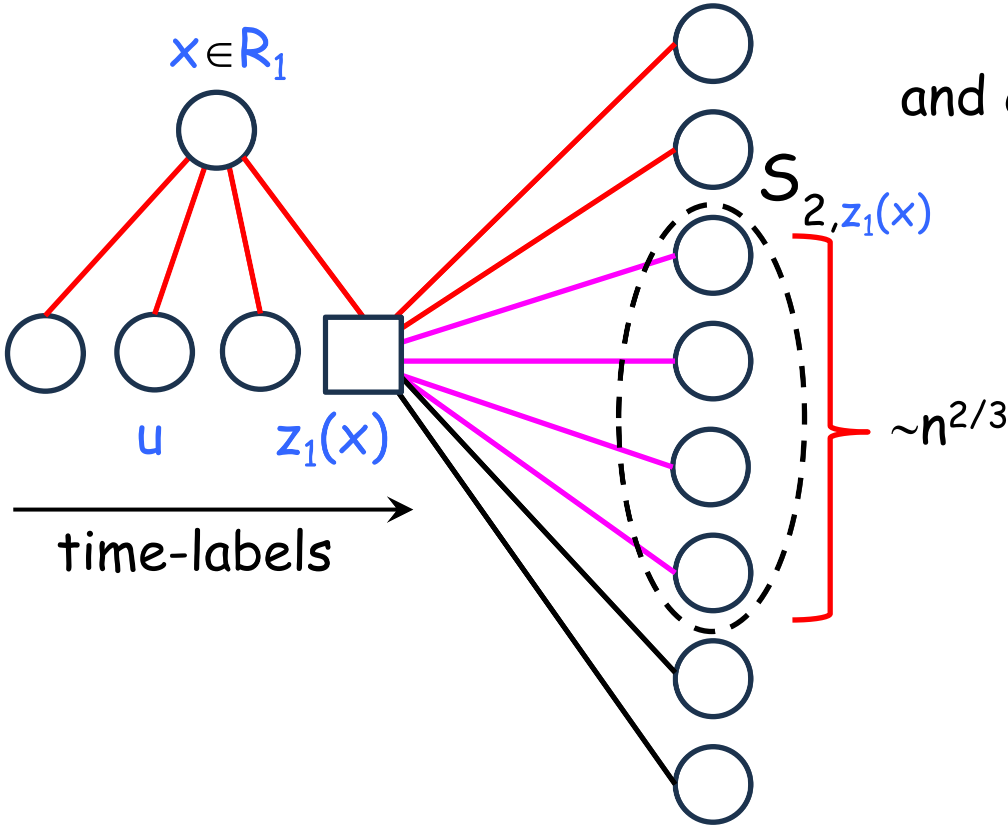


x ∈ R1
u
z1(x)
time-labels

compute a second-level hitting set R_2 of

$$\{ S_{2,z} \}_{z \in Z}$$

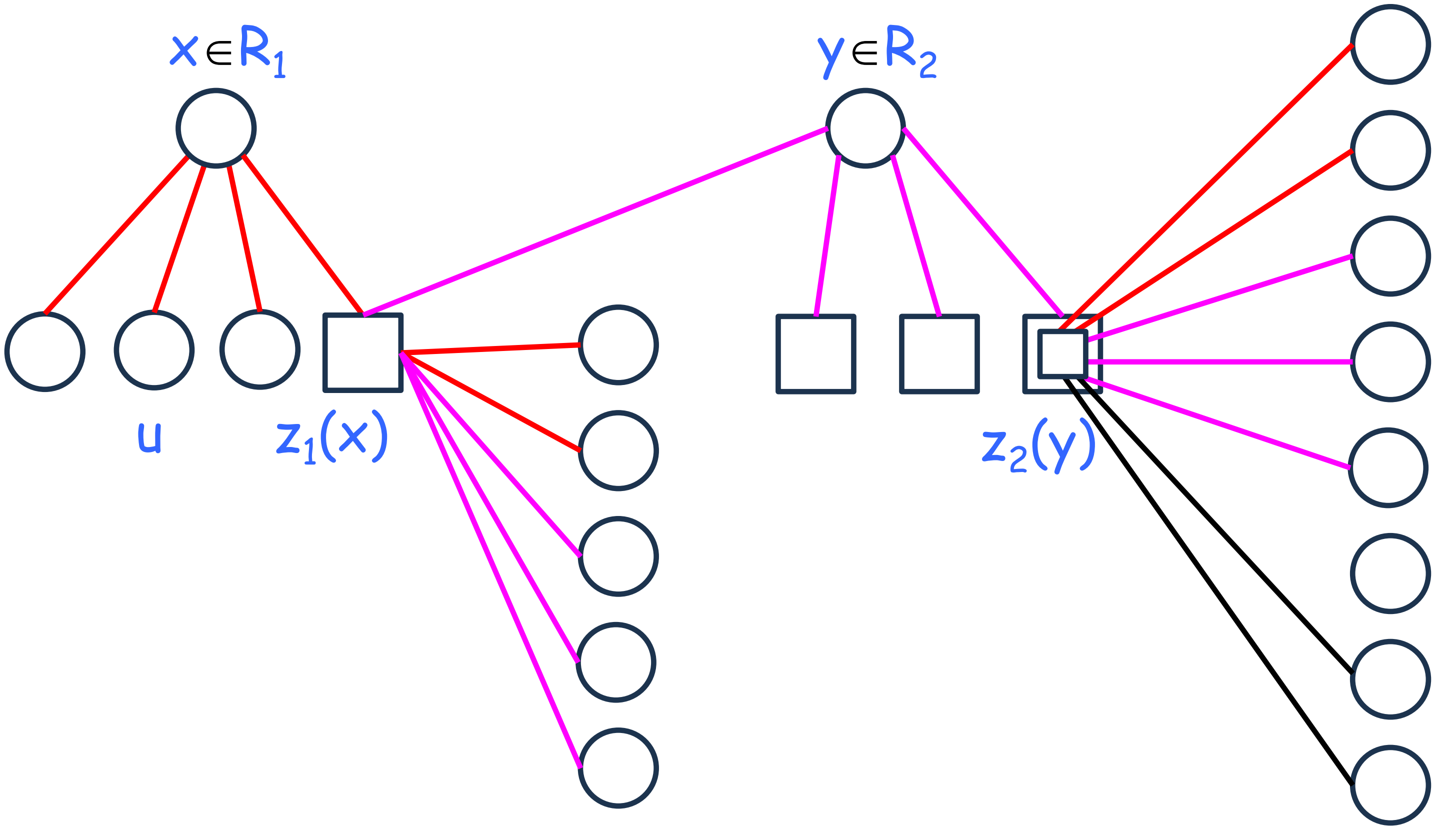
and cluster special vertices around R_2



H= red edges + purple edges

- \# red + #purple edges $\tilde{O}(n^{4/3})$

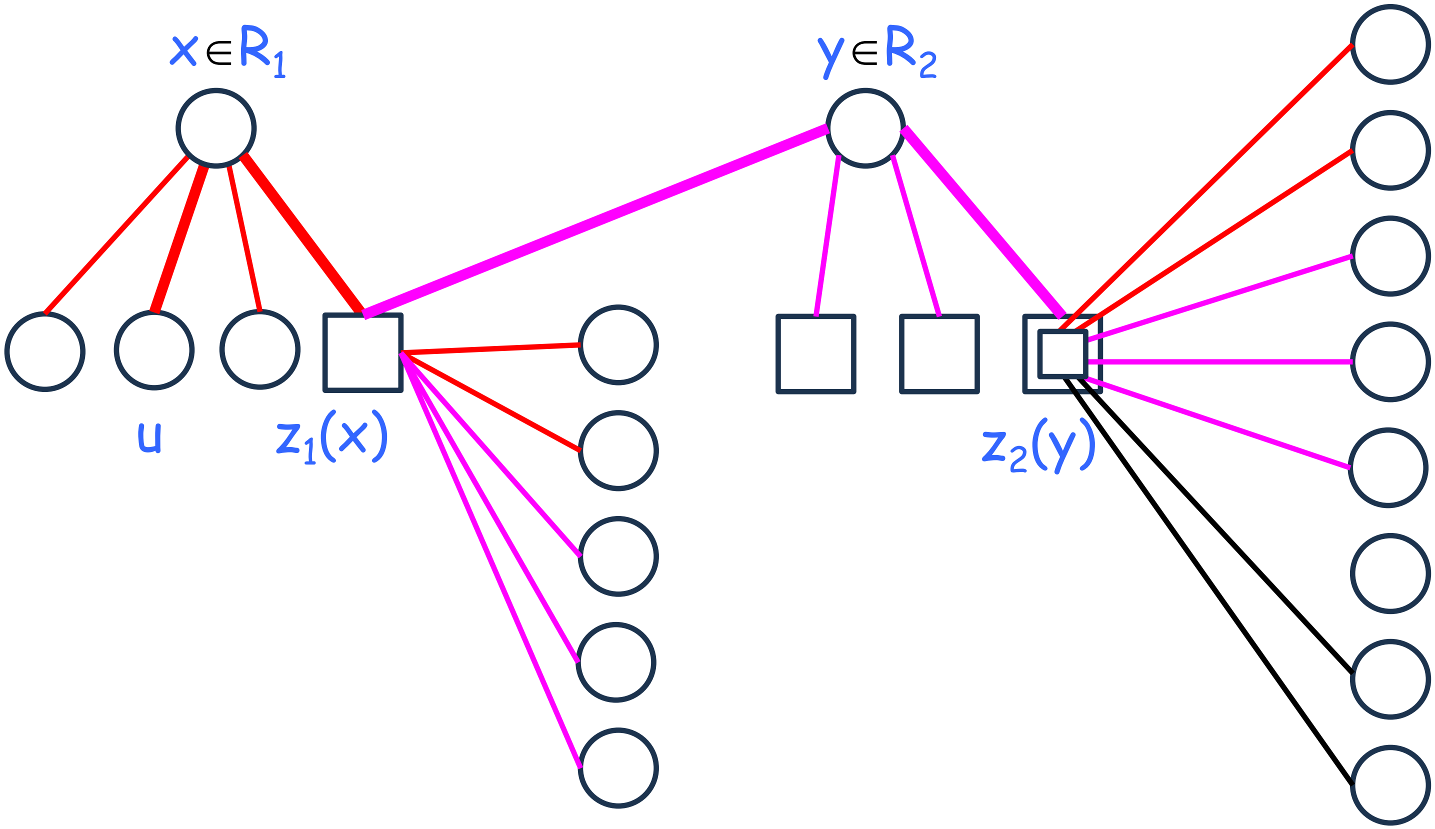
H = red edges + purple edges



for any $v \in V$

case: $v = z_1(x)$ or $v = z_2(y)$

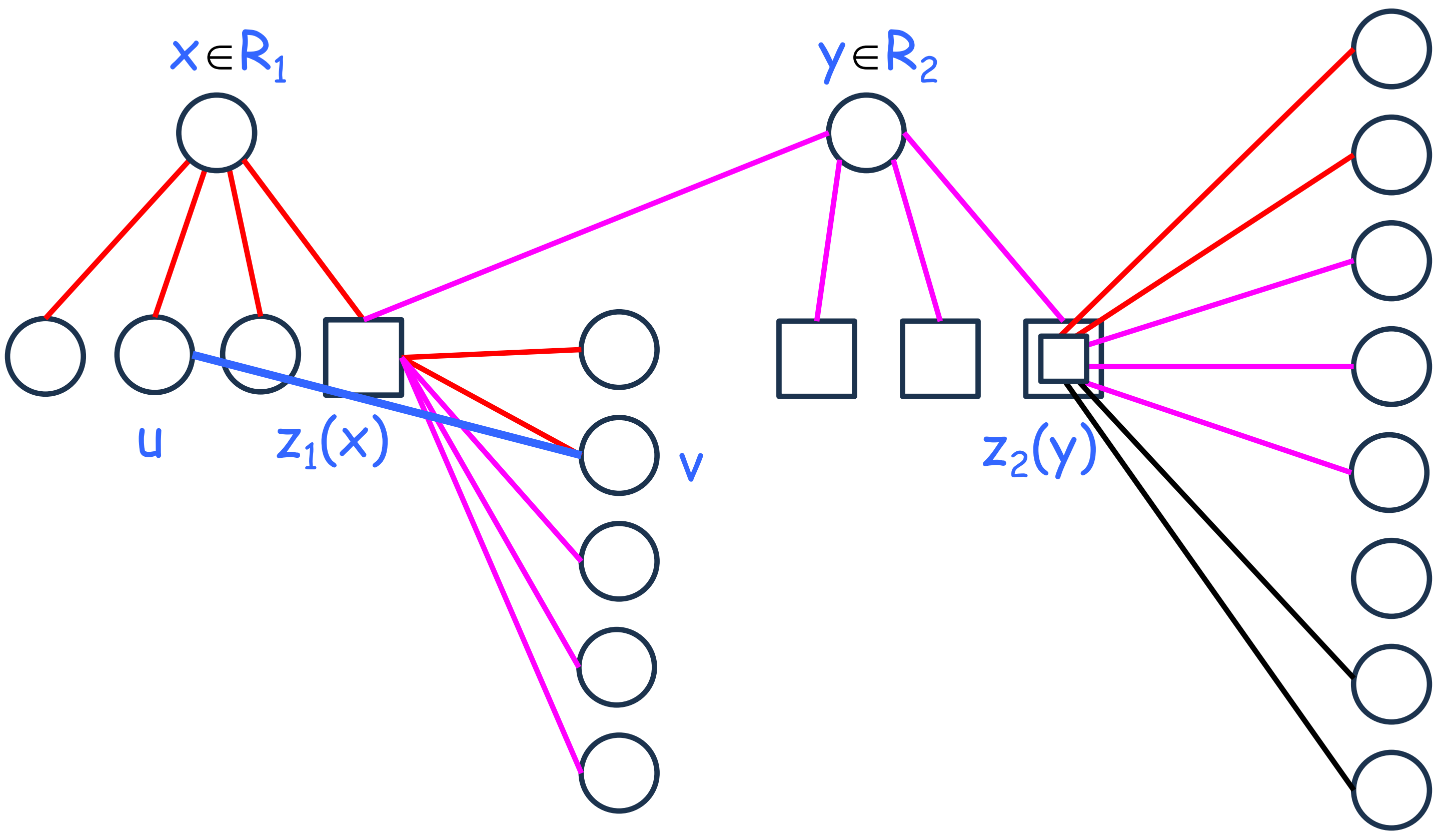
H= red edges + purple edges



for any $v \in V$

case: $v = z_1(x)$ or $v = z_2(y)$

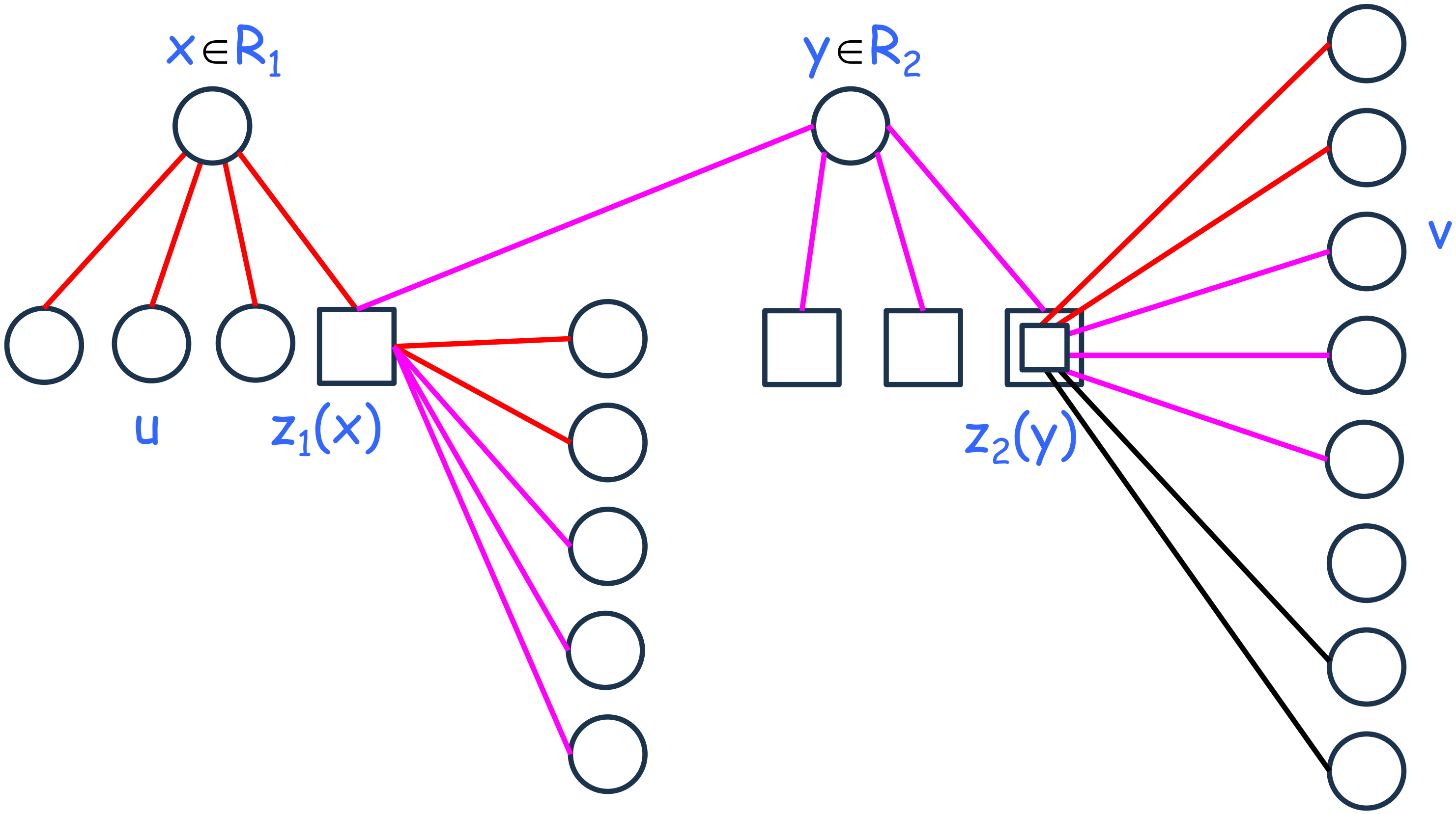
H= red edges + purple edges



for any $v \in V$

case: $v \in S_{1,z_1(x)}$

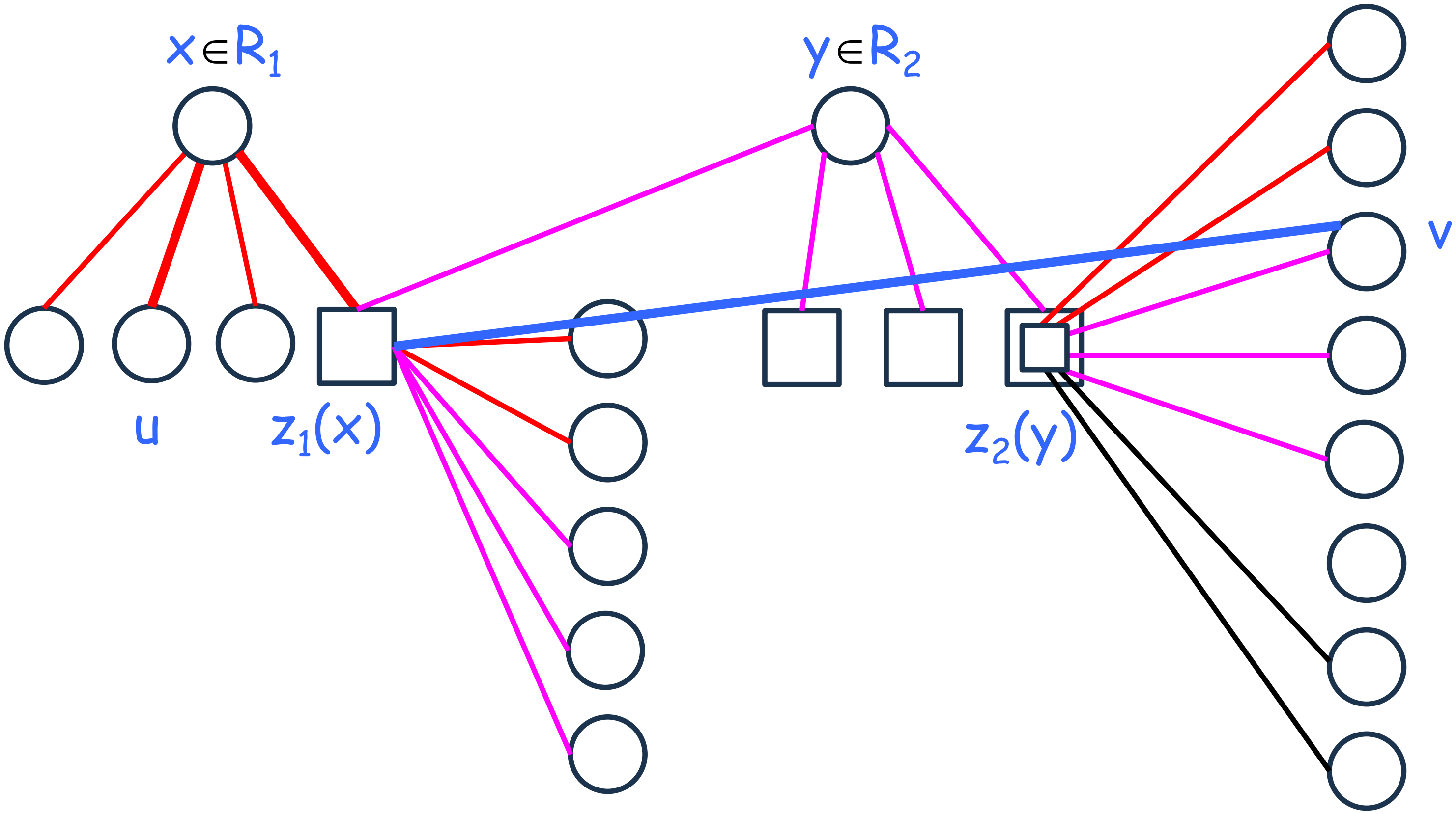
H = red edges + purple edges



for any $v \in V$

case: $v \in S_{1,z_2(y)}$ or $v \in S_{2,z_2(y)}$

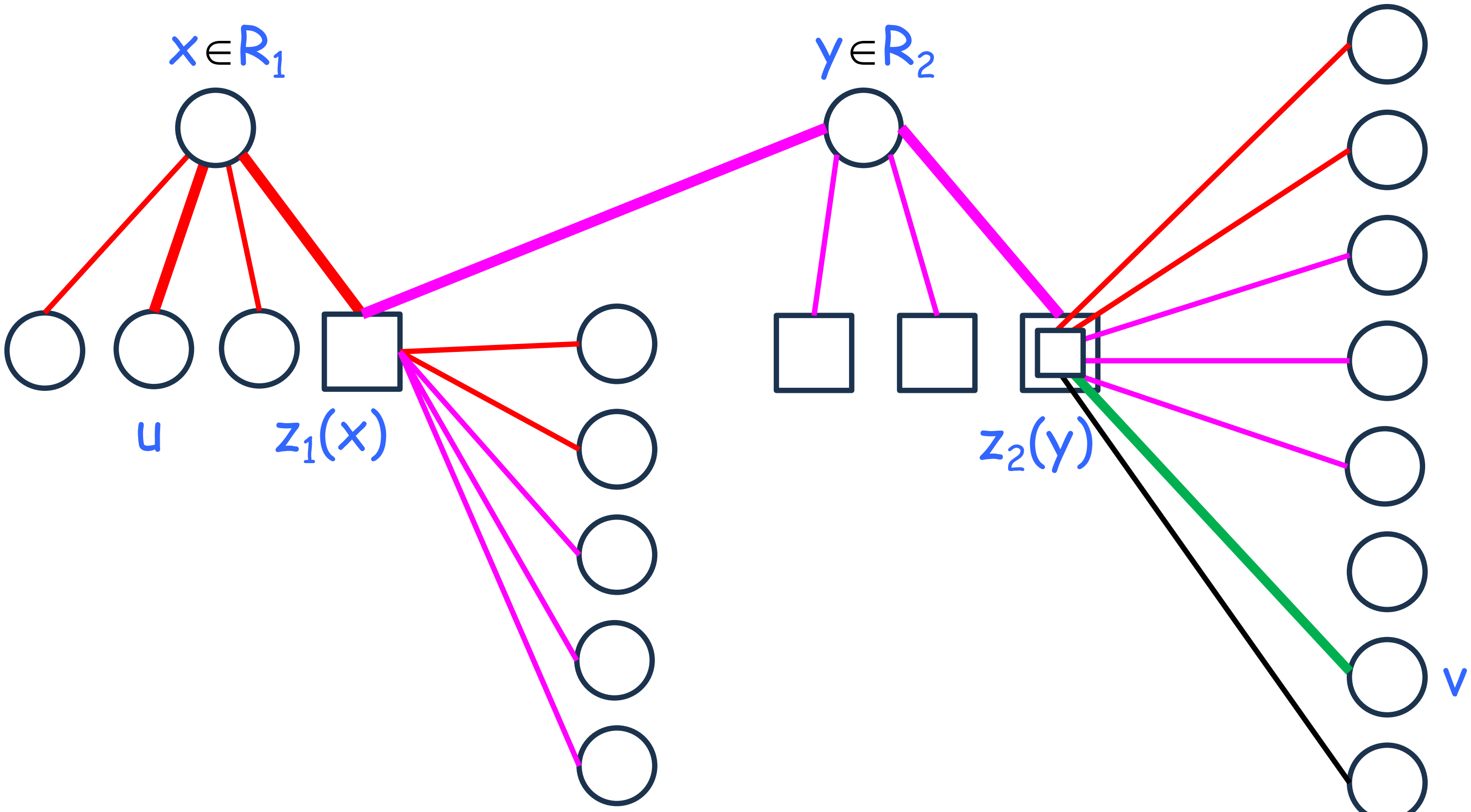
H= red edges + purple edges + blue edges



for any $v \in V$

case: $v \in S_{1,z_2(y)}$ or $v \in S_{2,z_2(y)}$

H= red edges + purple edges + blue edges + green edges



for any $v \in V$

case: $v \notin S_{1,z_2(y)}$ or $v \notin S_{2,z_2(y)}$

H is a 5-spanner of size $\tilde{O}(n^{4/3})$

Selected open problems

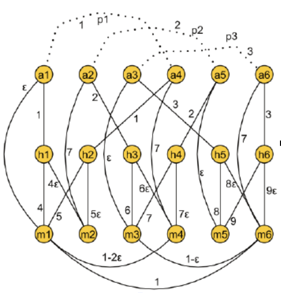
Cliques

3-spanner

$\Omega(n^{1+\varepsilon})$ vs $\tilde{O}(n)$

Beyond Cliques

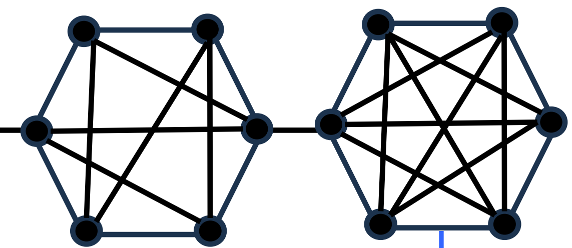
(dense)
general graphs



???

(n-2)-regular graph

cliques



$\Omega(n^2)$ for connectivity

for connectivity and logarithmic stretch $\tilde{O}(n)$

for stretch 2k-1 $\tilde{O}(n^{1+1/k})$

Thanks for your attention!