Temporal Menger and Related Problems

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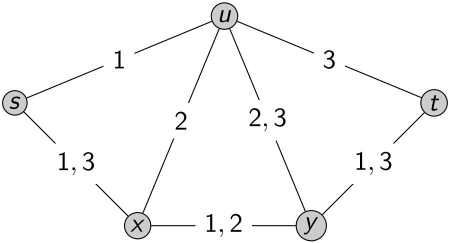
Joint work with A. Ibiapina, R. Lopes and A. Marino.

Basic definitions.

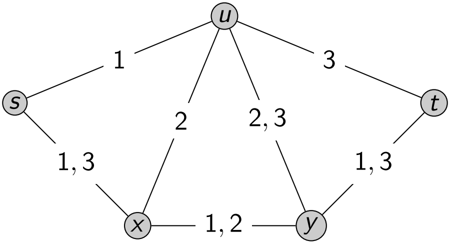
Temporal Graph



Definitions: Temporal Graph

- A temporal graph is a pair (G, λ) where G is a simple graph, and $\lambda : E(G) \rightarrow 2^{\mathbb{N}}$;

Temporal Graph

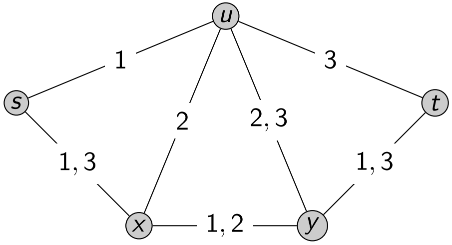


$\tau = 3$

Definitions: Lifetime

- A *temporal graph* is a pair (G, λ) where G is a simple graph, and $\lambda : E(G) \to 2^{\mathbb{N}}$;
- The value $\max_{e \in E(G)} \lambda(e)$ is called the lifetime; will be denoted by τ;

Temporal Graph



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Definitions: Temporal vertex/edge

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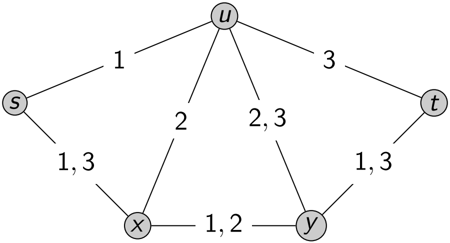
- The value $\max_{e \in E(G)} \lambda(e)$ is called the *lifetime*; will be denoted by τ;
- A pair (x, i) where $x \in V(G)$ and $i \in [\tau]$ is called a temporal vertex;

Temporal Graph



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Definitions: Temporal vertex/edge

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- The value $\max_{e \in E(G)} \lambda(e)$ is called the *lifetime*; will be denoted by τ;
- A pair (x, i) where $x \in V(G)$ and $i \in [\tau]$ is called a temporal vertex; similarly (e, i) s.t. $e \in E(G)$ and $i \in \lambda(e)$ is called a temporal edge.

Natural questions

What are the natural adaptations of basic Graph Theory concepts?

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Which structural/polynomial results on basic Graph Theory carry over?

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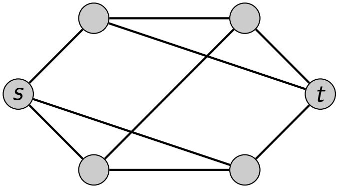
What are the natural adaptations of Menger's Theorem related concepts?

What versions of Menger's Theorem hold?
What are the complexities of the related problems?

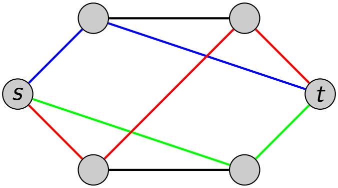
Disjoint paths and separators



Definitions: Disjoint s, t-paths

- A set of s, t-paths which are *internally vertex disjoint*;

Disjoint paths and separators



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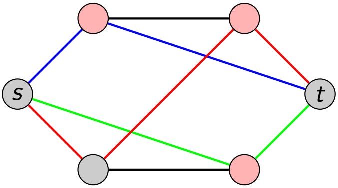

Definitions: s, t-separator

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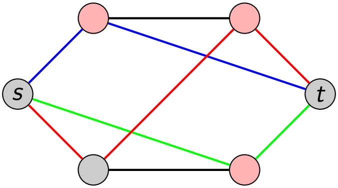


Definitions:

- A set of s, t-paths which are *internally vertex disjoint*; Maximize
- A subset $S \subseteq V(G) \setminus \{s, t\}$ s.t. every s, t-path intersects S. Minimize

Menger's Theorem

Theorem (Menger, 1927)

Given a graph G, and vertices $s, t \in V(G)$, then

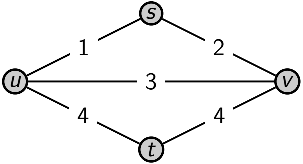
the maximum number of disjoint s, t-paths in G
is equal to
the minimum size of an s, t-separator.

These can also be found in polynomial time.

Karl Menger.
"Zur allgemeinen kurventheorie"
Fundamenta Mathematicae 10: 96–115, 1927.

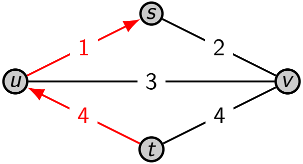
Menger's Theorem on Temporal Graphs.

Temporal walk/path



A walk/path that respects the flow of time.

Temporal walk/path



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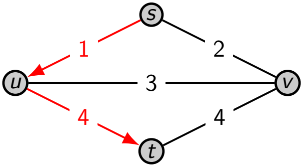

A walk/path that respects the flow of time (not necessarily strictly increasing).

Temporal Menger - Vertex disjoint version

Definitions: Disjoint temporal s, t-paths

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Definitions: temporal s, t-separator

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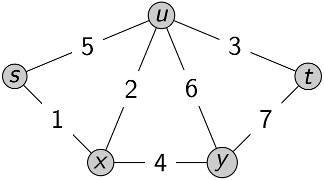
Vertex disjoint Menger's Property

Given a temporal graph (G, λ), and vertices $s, t \in V(G)$, then

the maximum number of disjoint *temporal s, t-paths*
is equal to
the minimum size of a temporal s, t-separator.

Vertex Disjoint Menger's Property does not hold



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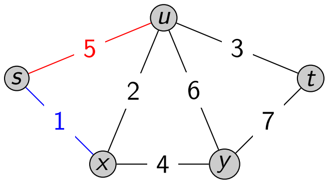

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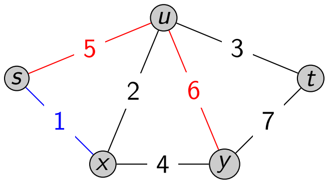

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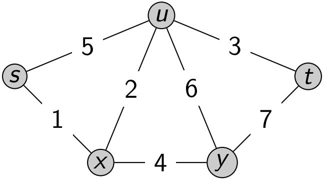

Maximum number of temporal s, t-paths is 1

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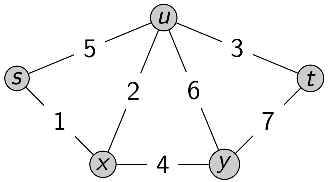

Maximum number of temporal s, t-paths is 1
Minimum size of a temporal s, t-separator is 2.

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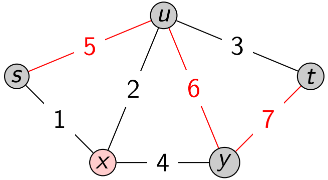

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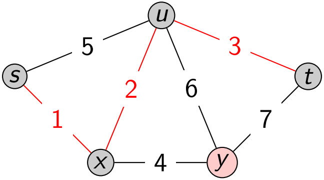

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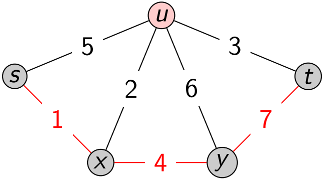

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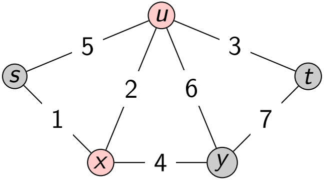

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Mengerian graphs

A graph G is Mengerian if equality always holds
(i.e., holds for every pair s, t and every possible time assignment on $E(G)$).

David Kempe, Jon Kleinberg and Amit Kumar.
"Connectivity and inference problems for temporal networks."
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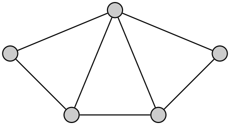

Theorem (Kempe, Kleinberg and Kumar, 2000)

If $|\lambda(e)| = 1$ for all edges, then
a graph G is Mengerian iff G has no gem subdivision.
It can be recognized in poly-time.

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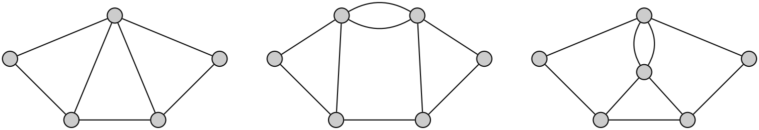
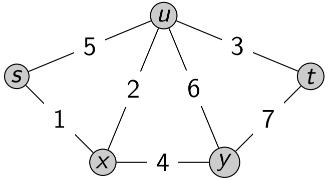


Theorem (Ibiapina and A.S.)

G is Mengerian iff G has none of the graphs above as m-subdivision.
It can be recognized in poly-time.

Allen Ibiapina and A.S..
"Mengerian Graphs Revisited."
FCT 2021.

Temporal vertex disjoint walks



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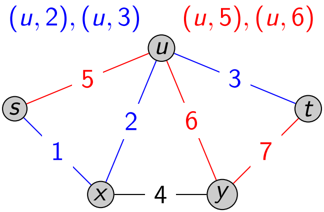

Definitions: t-vertex disjoint

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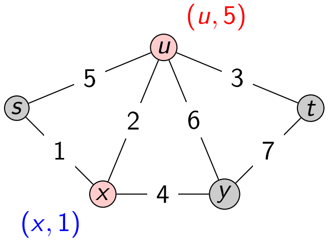
Temporal vertex disjoint walks



Definitions: t-vertex s, t-separator

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- A subset $S \subseteq (V(G) \setminus \{s, t\}) \times [\tau]$ s.t. every temporal s, t-walk intersects S.

Menger holds for t-vertex disjoint *walks*

Theorem (Mertzios, Michail and Spirakis, 2019.)

Given a temporal graph (G, λ), and vertices $s, t \in V(G)$, then

the maximum number of t-vertex disjoint temp. s, t-walks in $\mathcal{G}$
is equal to
the minimum size of a t-vertex s, t-separator.

These can also be found in polynomial time.

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For t-vertex disjointness, walks and paths differ.

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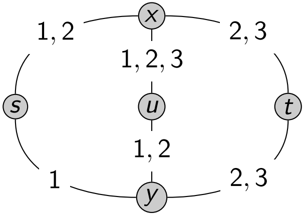
Menger *does not hold* for t-vertex disjoint *paths*



Does not have 3 t-vertex disjoint temp. s, t-paths (not walks).

Joint work with Ibiapina, Lopes and Marino.

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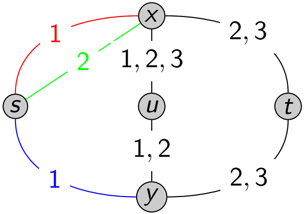

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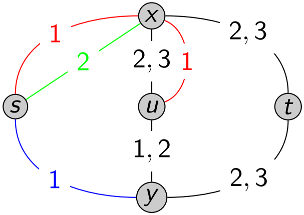

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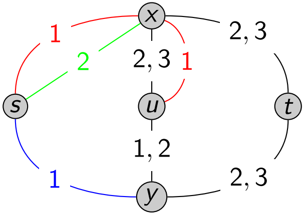

intersect in $(y, 1)$ or $(y, 2)$.

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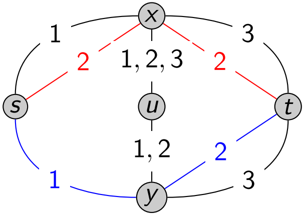

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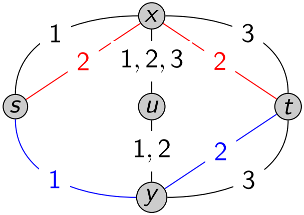

Maximum number of t-vertex disjoint temp. s, t-paths is 2.
Minimum size of $S \subseteq (V(G) \setminus \{s, t\}) \times [\tau]$ that intersects every temporal s, t-path is 3.

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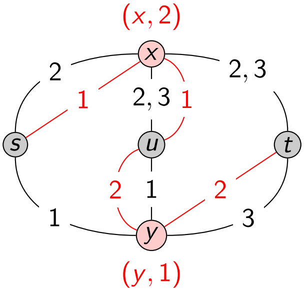
Menger *does not hold* for t-vertex disjoint *paths*



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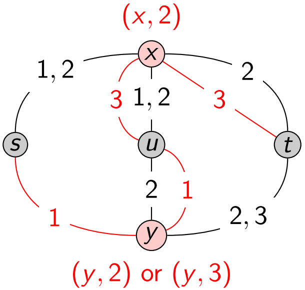

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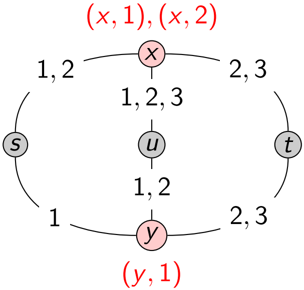

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Joint work with Ibiapina, Lopes and Marino.

Menger *holds* for t-vertex disjoint *paths* iff max paths= 1

T-Vertex Disjoint Menger's Property

maximum number k of t-vertex disjoint temp. s, t-paths in $\mathcal{G}$
is equal to
the minimum size of of $S \subseteq (V(G) \setminus \{s, t\}) \times [\tau]$ that
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T-Vertex Disjoint Menger's Property

maximum number k of t-vertex disjoint temp. s, t-paths in $\mathcal{G}$

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Theorem (Ibiapina, Lopes, Marino and **S.**.)

T-Vertex Disjoint Menger's Property always holds

(i.e. holds for every temporal graph $\mathcal{G} = (G, \lambda)$ and $s, t \in V(G)$)

iff

$k = 1$.

Complexity of related problems

Breaking temporal *paths* with temporal vertices

- For each $S \subseteq (V(G) \setminus \{s, t\}) \times [\tau]$ of size h

Breaking temporal *paths* with temporal vertices

- For each $S \subseteq (V(G) \setminus \{s, t\}) \times [\tau]$ of size h
  - Test whether S intersects all temporal s, t-paths

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Theorem (Ibiapina, Lopes, Marino and S..)

Testing this is co-NP-complete.

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Testing this is co-NP-complete.

Theorem (Ibiapina, Lopes, Marino and S..)

But it can be done in time $O(h^h m\tau)$.

Finding 2 t-vertex disjoint *paths*

(T) Testing whether there exists $(u, i) \in (V(G) \setminus \{s, t\}) \times [\tau]$ that intersects all temporal s, t-paths can be done in time $O(mn\tau^2)$.

Finds 2 t-vertex disjoint s, t-*paths*.

1. If test (T), return No.

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2. Let $\mathcal{G}' = \mathcal{G}$.
3. For each temporal edge $(e, i) \in E(G) \times [\tau]$.
   - If (T) applied to $\mathcal{G}' - (e, i)$ does not hold, then remove (e, i).

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4. Do a search to find the 2 t-vertex disjoint paths in $\mathcal{G}'$ and return.

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Finds 2 t-vertex disjoint s, t*-paths* in time $O(m^2 n\tau^3)$.

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Finding 2 t-vertex disjoint *paths*

(T) Testing whether there exists $(u, i) \in (V(G) \setminus \{s, t\}) \times [\tau]$ that intersects all temporal s, t-paths can be done in time $O(mn\tau^2)$.

Finds 2 t-vertex disjoint s, t*-paths* in time $O(m^2 n\tau^3)$.

1. If test (T), return No.
2. Let $\mathcal{G}' = \mathcal{G}$.
3. For each temporal edge $(e, i) \in E(G) \times [\tau]$.
   - If (T) applied to $\mathcal{G}' - (e, i)$ does not hold, then remove (e, i).
4. Do a search to find the 2 t-vertex disjoint paths in $\mathcal{G}'$ and return.

Size of a separator is 1 when max-paths is 1 (Menger holds).

Related Problems

		$\geq k$ Disjoint Paths/Walks	Menger's Th.	$\leq h$ Separator
vertex	walk/ path			
t-vertex	walk			
	path			

Complexity of related problems

		$\geq k$ Disjoint Paths/Walks	Menger's Th.	$\leq h$ Separator
vertex	walk/ path	NP-c even if, G dir., $\tau = k = 2$ [1] or G undir., $k = 2$ [2]	$\neq$ [1]	NP-c even if $\tau = 2$, W[1] for h [3]
t-vertex	walk	Poly [4]	$=$ [4]	Poly [4]
t-vertex	path	Poly if $k = 1$ [5]	$=$	Poly if $h = 1$
		Poly if $k = 2$	$\neq$	Poly if $h = 2$
		NP-c if G undirected, or if G directed, even if $\tau = 3$ for every fixed $k \geq 3$	$\neq$	co-NP-hard for given h XP for h

Allen Ibiapina, Raul Lopes, Andrea Marino and **A.S.**.
Menger's Theorem for Temporal Paths (Not Walks).
arXiv:2206.15251.

Bibliography

[1] Kenneth A. Berman
"Vulnerability of scheduled networks and a generalization of Menger's Theorem".
Networks 28 (3) (1996) 125–134.

[2] David Kempe, Jon Kleinberg and Amit Kumar.
"Connectivity and inference problems for temporal networks."
STOC'00.

[3] Philipp Zschoche, Till Fluschnik, Hendrik Molter and Rolf Niedermeier,
"The complexity of finding small separators in temporal graphs."
J. Computer System and Sciences 107 (2020) 72–92.

[4] George B. Mertzios, Othon Michail and Paul Spirakis.
" Temporal network optimization subject to connectivity constraints. "
Algorithmica 81 (4): 1416–1444, 2019.

[5] Huanhuan Wu, James Cheng, Yiping Ke, Silu Huang, Yuzhen Huang, and Hejun Wu,
"Efficient algorithms for temporal path computation."
IEEE Trans. on Knowl. and Data Engin. 28(11) (2016) 2927–2942.

Final remarks and open questions

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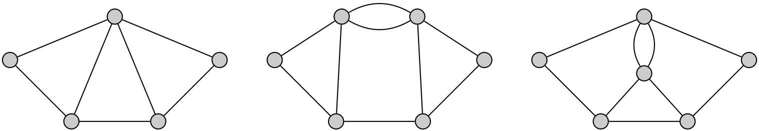
- Structural properties can lead to poly algorithms;
- Other temporal version of Menger's Theorem: node departure time (Mertzios, Michail and Spirakis, 2019 [4]) - closer to edge versions of Menger.

Open questions



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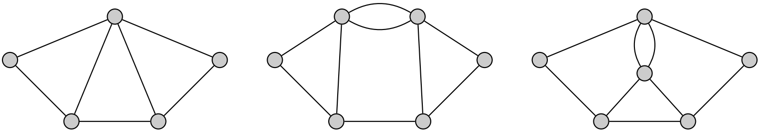

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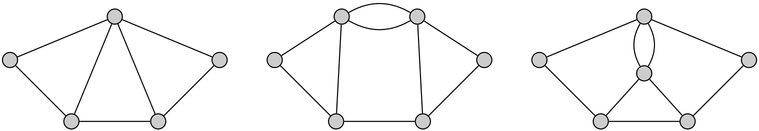

- Given a simple graph G with no gem as topological minor, can we assign λ to G in order to ensure Menger's Property?
- Can we find k t-vertex disjoint paths in poly time if k is fixed and G is undirected?
- Are there other uncovered "temporal Menger" concepts?

Muito obrigada!
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