Sharp Thresholds in Random Simple Temporal Graphs

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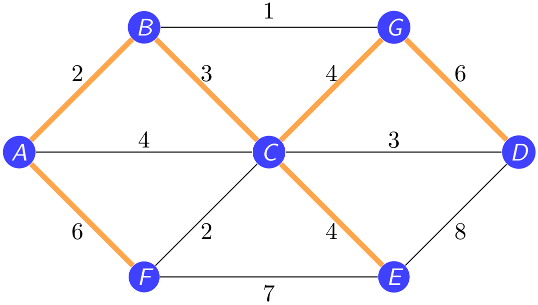
Temporal graphs and paths



Temporal graph: graph with edge presence times

Temporal path: path with edges crossed at increasing presence times

Graph properties

Our focus: connectivity

- Multiple facets in the temporal setting
- No guarantee that one source-sink is enough
- No guarantees that spanners are small
- Bad examples — exceptional or typical?

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Connectivity, static graphs

- Transitive
- Directed graph strongly connected iff some source is a sink
- Connectivity always demonstrated by a linear subgraph
  (incoming + outgoing tree)
  Single spanning tree in undirected case
- Easy to check

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Connectivity, temporal graph
Worst-case

- Temporal connectivity without $O(n)$-spanners
  [Kempe, Kleinberg, Kumar 2000]
- Temporal connectivity by $\Theta(n^2)$ labels without $o(n^2)$-spanners
  [Axiotis, Fotakis 2016]
- Ordered cliques have $O(n \log n)$-spanners
  [Casteigts, Peters, Schoeters 2021]
- Minimal spanner: APX-hard
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Stochastic question

Things can be complicated!

...and hard to check

Question: are these behaviours exceptional or typical?

Connectivity in random temporal settings

- Similar random temporal settings
  Call-any gossiping, Rendezvous population protocols...
- Future interactions independent from past
- Connectivity happens soon
  ($\approx 1.5n \log n$ edges expected value)
- Spanners?
- Simple temporal graphs: no repetitions — implications?

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Random *Simple* Temporal Graphs

- Simple: each edge has one label
  Labels are distinct
- Main model we study: RSTG
- Equivalent representations
  - $\mathcal{G}_{n,p}$ (Erdős–Rényi random graph) with random edge order
  - $\mathcal{G}_{n,p}$ with random edge labels from $[0;1]$
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  - K_n (clique) random edge labels from $[0;1]$, edges after p removed
- Notation: $\mathcal{F}_{n,p}$
- When things happen as p grows for fixed labelling of K_n?

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RSTG vs. Call-any

- Similar behaviour

  Few edges get actually repeated
- But independence of past and future is destroyed
- Weak dependencies
- We handle these (now working to handle more)

Connectivity

- $\frac{1}{n}$: footprint gets linear-size connected component
- $\frac{\log n}{n}$: footprint gets connected
- $\frac{\log n}{n}$: most pairs get connectivity
- $\frac{\log n}{n}$: (current WIP) $n - o(n)$ vertices get pairwise connectivity
- $\frac{2 \log n}{n}$: first temporal source
- $\frac{2 \log n}{n}$: most vertices temporal sources
- $\frac{3 \log n}{n}$: all vertices temporal sources
  (graph is temporally connected)

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Connectivity: main points

- $\frac{\log n}{n}$: $\forall u : \forall v :$ **a.a.s.** $u \rightsquigarrow v$
- $\frac{2\log n}{n}$: $\forall u :$ **a.a.s.** $\forall v : u \rightsquigarrow v$
- $\frac{3\log n}{n}$: **a.a.s.** $\forall u : \forall v : u \rightsquigarrow v$

Spanners

Can connectivity be shown by a small subgraph?

- $\mathcal{F}_{n,(3+o(1))\frac{\log n}{n}}$ has $(1.5+o(1))n\log n$ edges...
- Can we go smaller?
- Need at least $2n-4$ labels [Bumby 1970]
- Optimal spanner exists in $\mathcal{F}_{n,(4+o(1))\frac{\log n}{n}}$

  Current work in progress: but not earlier
- $2n+o(n)$-spanner exists in $\mathcal{F}_{n,(3+o(1))\frac{\log n}{n}}$

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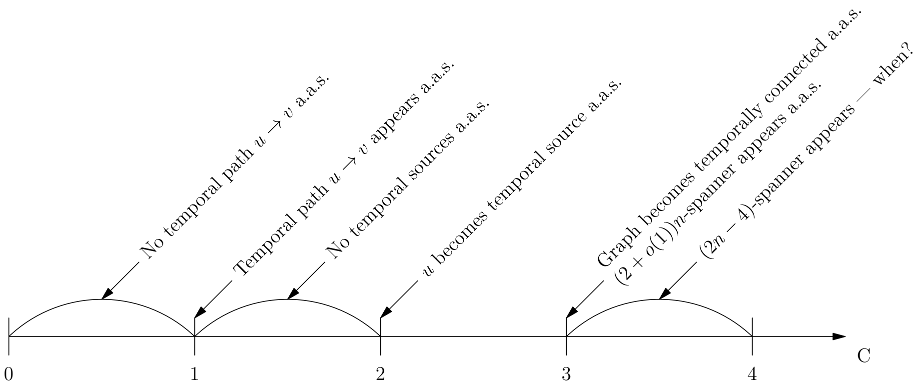
Spanners: main points

- $\frac{3\log n}{n}$: $2n + o(n)$-spanner
- $\frac{4\log n}{n}$: $2n - 4$-spanner

Timeline


No temporal path $u \to v$ a.a.s.
Temporal path $u \to v$ appears a.a.s.
No temporal sources a.a.s.
u becomes temporal source a.a.s.
Graph becomes temporally connected a.a.s.
$(2+o(1))n$-spanner appears a.a.s.
$(2n-4)$-spanner appears — when?
0
1
2
3
4
C

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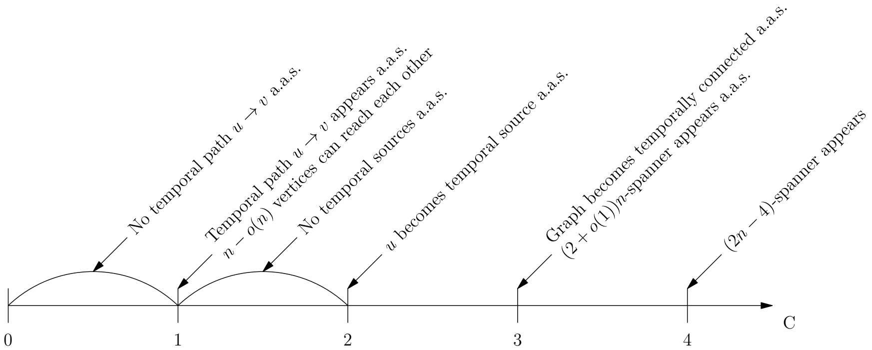

(With WIP results)

Structure of connectivity

- Normally takes $\frac{\log n}{n}$ time and $O(\log n)$ hops for $u \rightsquigarrow v$
- But a vertex can «sleep» with not edges for some time
- Longest observed sleeping time: $\frac{\log n}{n}$
- Sleep determines extra $\frac{\log n}{n}$ per shift of **a.a.s.**!

Construction: Foremost tree

Building block for larger construction

Tree of temporal paths

- Start with a single vertex

  advanced version: and some starting time

- Add earliest later edge that adds a new vertex to tree
- Repeat

Same can be done in reverse

Construction: Nearly-optimal spanner

- Temporal paths between all vertices
- *Short* temporal paths between all vertices
- Pivot vertex
- Reaches almost everything in $\approx 1\frac{\log n}{n}$,
  reaches everything in $\approx 2\frac{\log n}{n}$
- Two large trees of temporal paths:
  almost source from $\approx 2\frac{\log n}{n}$, almost sink by $\approx 1\frac{\log n}{n}$
- Earliest paths to pivot / latest paths from pivot for the rest
- Still not enough for full connectivity, connect directly
- Make sure this has $(2 + o(1))\frac{\log n}{n}$ edges

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Conclusion

- Obstacles to small spanners and early connectivity are rare
- In random simple temporal graphs nearly optimal spanners appear nearly immediately after connectivity
- Weak dependencies can be handled

Thanks for your attention!

Questions?

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